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FINITE VOLUME BOX SCHEMES AND MIXED METHODS

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Abstract. We present the numerical analysis on the Poisson problem of two mixed Petrov-Galerkin Finite Volume schemes for equations in divergence form $\mathrm{div}\,\varphi(u, \nabla u) = f$. The first scheme, which has been introduced in [22], is a generalization in two dimensions of Keller's box-scheme. The second scheme is the dual of the first one, and is a cell-centered scheme for u and the flux φ. For the first scheme, the two trial Finite Element spaces are the nonconforming space of Crouzeix-Raviart for the primal unknown u and the div-conforming space of Raviart-Thomas for the flux φ. The two test spaces are the functions constant per cell both for the conservative and for the flux equations. We prove an optimal second order error estimate for the box scheme and we emphasize the link between this scheme and the post-processing of Arnold and Brezzi of the classical mixed method.

Résumé. Nous effectuons l'analyse numérique pour le problème de Poisson de deux schémas volumes finis mixtes de type Petrov-Galerkin pour des équations sous forme divergence $\mathrm{div}\,\varphi(u, \nabla u) = f$. Le premier schéma, qui a été introduit dans [22], est une généralisation à deux dimensions du schéma boîte de Keller. Le second schéma, dual du premier, est de type "cell-center" pour u et pour le flux φ. Dans le premier schéma, les deux espaces d'approximation sont l'espace non conforme de Crouzeix-Raviart pour l'inconnue primale u et l'espace de Raviart-Thomas pour le flux φ. Les deux espaces test sont les espaces des fonctions constantes par cellule, à la fois pour l'équation conservative et l'équation du flux. Nous prouvons une estimation d'erreur optimale en $O(h^2)$ pour le schéma boîte et nous mettons en évidence le lien entre ce schéma et le post-processing d'Arnold et Brezzi de la méthode mixte classique.

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1. Introduction

In this paper, we perform the numerical analysis of two Finite Volume schemes for conservative equations. The first scheme is a *Finite Volume Box Scheme*, introduced in [22]. This kind of scheme originates in the pioneering paper by H.B. Keller [31], where the basic principles of box-schemes are introduced on the model problem of the 1D heat equation. In the lowest order version, these principles are, for a problem in divergence form like $\mathrm{div}\,\varphi(u, \nabla u) = f$, firstly to use degrees of freedom located on the faces of the mesh (the edges in two dimensions) both for the unknown u and the flux φ , secondly to build the discrete equations by averaging the continuous ones onto "boxes". In this sense, they are finite-volume schemes, that is, schemes ensuring a local conservation property at the level of the mesh.

Keywords and phrases: Box Method - Box Scheme - Mixed Finite Element Method - Petrov Galerkin Method - Finite Volume Method

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Beside a direct use of Keller's scheme for variants of 1D heat equations in some works, [27,34], two types of *box-schemes* are known in the literature.

The first one has been introduced in Computational Fluid Dynamics, especially the compressible Euler equations. The main idea is that locating the conservative unknowns (density, momentum, energy) at the center of the faces of the mesh is very natural in order to build a scheme discretizing both the conservative and the flux equations at the level of a single cell. We refer to [14–17,19–21,41,42] where numerical results on Euler or Navier-Stokes equations are presented.

Another kind of box scheme is known under the name of *Box Method* or *Finite Volume Element Method*. The design of this scheme is similar to the one of the so called *cell-vertex* Finite-Volume method, (see e.g. [28]). The unknown u belongs to a Finite Element space such that P^1 or Q^1. The discrete equations are defined from averaging the continuous ones onto a dual box surrounding each vertex. We refer to [4,13,30,32,37–39] and the references therein.

The scheme introduced in [22] for the 2D Poisson problem on a triangular mesh is a Keller-like scheme. Contrary to [15, 16, 19] where a Finite Differencing interpretation of the discrete unknowns is used, a Finite Element interepretation of the degrees of freedom is introduced. On a regular FEM triangulation $\mathcal{T}_h$ by triangles K, the mixed form of the Poisson problem

$$\left\{\begin{array}{c} \operatorname{div} p + f = 0 \ \text{ in } \Omega \\ p - \nabla u = 0 \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega \end{array}\right. \tag{1}$$

is approximated by the finite volume scheme (called *FVbox* in the sequel): find $(u_h\,,\,p_h) \in P^1_{nc,0} \times RT^0$ such that

$$\left\{\begin{array}{cc} \langle \operatorname{div} p_h + f, \mathbb{1}_K\rangle = 0 & \forall\ K \in \mathcal{T}_h \\ \langle p_h - \nabla u_h, \mathbb{1}_K\rangle = 0 & \forall\ K \in \mathcal{T}_h \\ u_h = 0 & \text{on } \partial\Omega \end{array}\right. \tag{2}$$

In (2), $P^1_{nc,0}$ is the nonconforming space of Crouzeix-Raviart, [24], with homogeneous boundary conditions, and RT^0 is the div-conforming space of Raviart-Thomas of lowest order. Note that the coupling between these two spaces is unusual in the classical variational mixed methods theory, [8, 9, 11, 29], because this couple of spaces does not satisfy the Babuška-Brezzi condition.

The aim of this paper is to prove that the numerical analysis of (2) can be simply performed by using the classical theory of the mixed Petrov-Galerkin approximations, [6,7,35]. For simplicity of the notation, we restrict ourselves here to the academic Poisson problem in two dimensions, but generalizations to the 3D case or to more complex conservative problems are possible.

The outline of the paper is as follows. In Sect.2 we introduce the mixed formulation of which the *FVbox* scheme is an approximation. In fact, this formulation generates also a second scheme, (called *dual FVbox*), which is a Cell-Centered Finite Volume scheme for the couple of unknowns $(u, \nabla u)$, and which is also apparently new. We verify that the theory of nonconforming mixed approximations applies to the two schemes. A second order estimate is derived in Sect.2.4 for the *FVbox* scheme, using an Aubin-Nitsche argument. Finally, we prove in Sect.3 that the *FVbox* scheme coincides in fact with one of the post-processings of Arnold and Brezzi of the classical mixed method, [2], or equivalently, with the a posteriori interpretation of Marini, [33].

Let us mention finally that several recent papers deal with the numerical analysis or the design of Cell-Centered Finite Volume methods with the help of the Mixed Finite Element theory. A first kind of works, [5,25,43] is devoted to the a posteriori interpretation of the standard mixed solution u_h as a cell-centered Finite Volume Method. In [40], a new Mixed Finite Volume Method, different from the present one, is introduced and analyzed by the mixed Petrov-Galerkin theory. Finally, for a numerical analysis of the cell-centered Finite Volume method without reference to the FEM analogy, we refer to the exhaustive study [26]. See also [18].

This work was announced in [23].

2. Numerical analysis of the FVbox scheme

2.1. Two mixed forms of the Poisson problem

We consider the bidimensional Poisson problem in mixed form in a bounded domain $\Omega \subset \mathbb{R}^2$

$$(M) \quad \left\{ \begin{array}{c} \operatorname{div} p + f = 0 \ \text{ in } \Omega \\ p - \nabla u = 0 \ \text{ in } \Omega \\ u = 0 \ \text{ onto } \partial\Omega \end{array} \right. \tag{3}$$

Let us recall some standard notation. We denote by L^2 the Hilbert space of square integrable functions on Ω, equipped with the norm $|u|_{0,\Omega}$. H^1, H^2 are the standard Sobolev spaces equipped with their norms $\|u\|_1 = (|u|_{0,\Omega}^2 + |\nabla u|_{0,\Omega}^2)^{1/2}$ and $\|u\|_2 = (\|u\|_{1,\Omega}^2 + |D^2 u|_{0,\Omega}^2)^{1/2}$. H_0^1 is the Hilbert space of functions $u \in H^1$ having a null trace onto $\partial\Omega$. The semi-norm $|u|_{1,\Omega} = |\nabla u|_{0,\Omega}$ is a norm on H_0^1 equivalent to $\|u\|_1$. Finally H_{div} is the Hilbert space of bidimensional vector fields $p \in (L^2)^2$ and $\operatorname{div} p \in L^2$, equipped with the norm $\|p\|_{\text{div},\Omega} = (|p|_{0,\Omega}^2 + |\operatorname{div} p|_{0,\Omega}^2)^{1/2}$.
If $f \in L^2$ and Ω is C^2 or convex, this problem has a unique solution $u \in H_0^1 \cap H^2$ which verifies $\|u\|_{2,\Omega} \leq C|f|_{0,\Omega}$.
We do not address here the well-known mixed formulations with unknowns $(u,p) \in H_0^1 \times (L^2)^2$ or $(u,p) \in L^2 \times H_{\text{div}}$, wich are the continuous framework respectively of the classical conforming (or nonconforming) method and of the mixed method of Raviart and Thomas.

The two mixed formulations that are needed in the sequel are of Petrov-Galerkin type, that is, they use two different Hilbert spaces as primal and dual spaces. The first one is: find $(u,p) \in H_0^1 \times H_{\text{div}}$ such that

$$(P_1) \quad \left\{ \begin{array}{ll} (\operatorname{div} p + f, v)_{0,\Omega} = 0 & \forall v \in L^2 \\ (p - \nabla u\,,\, q)_{0,\Omega} = 0 & \forall q \in (L^2)^2 \end{array} \right. . \tag{4}$$

Note that this formulation is also introduced in [40]. The second-one is the dual of (P_1). It reads : find $(v,q) \in L^2 \times (L^2)^2$ such that

$$(P_2) \quad \left\{ \begin{array}{ll} -(\nabla u\,,\, q)_{0,\Omega} + (f,u)_{0,\Omega} = 0 & \forall u \in H_0^1 \\ (p,q)_{0,\Omega} + (\operatorname{div} p, v)_{0,\Omega} = 0 & \forall p \in H_{\text{div}} \end{array} \right. . \tag{5}$$

These two formulations are connected with the bilinear continuous form $B[(u,p)\,;\,(v,q)]$ defined for $(u,p) \in H_1 = H_0^1 \times H_{\text{div}}$, $(v,q) \in H_2 = L^2 \times (L^2)^2$ by

$$B[(u,p)\,;\,(v,q)] = a\,(p,q) + b_1\,(q,u) + b_2\,(p,v) \tag{6}$$

where a is the continuous bilinear form defined on $H_{\text{div}} \times (L^2)^2$ by

$$a\,(p,q) = (p,q)_{0,\Omega} \tag{7}$$

and b_1 , b_2 are the two forms respectively defined for $(u,q) \in H_0^1 \times (L^2)^2$ and $(v,p) \in L^2 \times H_{\text{div}}$ by

$$b_1\,(q,u) = -(\nabla u\,,\, q)_{0,\Omega} \quad , \; b_2(p,v) = (\operatorname{div}\, p\,,\, v)_{0,\Omega}. \tag{8}$$

We recall now briefly the following abstract result due to Babuška [3], Brezzi [10], Nicolaides [35], Bernardi *et al.* [6,7], devoted to the abstract formulation of mixed problems. Suppose given four Hilbert spaces

$$(X_1\,,\,(\cdot\,,\,\cdot)_{X_1})\,,\,(X_2\,,\,(\cdot\,,\,\cdot)_{X_2})\,,\,(M_1\,,\,(\cdot\,,\,\cdot)_{M_1})\,,\,(M_2\,,\,(\cdot\,,\,\cdot)_{M_2}) \tag{9}$$

and
(a) a continuous bilinear form $a\,(\cdot\,,\,\cdot) : X_1 \times X_2 \to \mathbb{R}$.

(b) two continuous bilinear forms $b_1(\cdot,\cdot)\colon X_2 \times M_1 \to \mathbb{R}$, $b_2(\cdot,\cdot)\colon X_1 \times M_2 \to \mathbb{R}$. We call H_1 , H_2, the Hilbert spaces $H_1 = X_1 \times M_1$, $H_2 = X_2 \times M_2$ equipped with the norms

$$\|(u,\lambda)\|_{H_1} = (\|u\|_{X_1}^2 + \|\lambda\|_{M_1}^2)^{1/2} \tag{10}$$

$$\|(v,\mu)\|_{H_2} = (\|v\|_{X_2}^2 + \|\mu\|_{M_2}^2)^{1/2}. \tag{11}$$

The null spaces $V_1 \subset X_1$, $V_2 \subset X_2$ are defined by

$$V_1 = \{u \in X_1 \mid b_2(u,\mu) = 0 \qquad \forall\, \mu \in M_2\} \tag{12}$$

$$V_2 = \{v \in X_2 \mid b_1(v,\lambda) = 0 \qquad \forall\, \lambda \in M_1\}. \tag{13}$$

For any $l_2 \in X_2'$, $m_2 \in M_2'$, we consider the abstract problem *(M)*: find $(u,\lambda) \in X_1 \times M_1$ such that

$$(M) \quad \left\{ \begin{array}{ccll} a(u,v) + b_1(v,\lambda) & = & < l_2\,,\, v >_{X_2',\,X_2} & \forall\, v \in X_2 \\ b_2(u,\mu) & = & < m_2\,,\, \mu >_{M_2',\,M_2} & \forall\, \mu \in M_2. \end{array} \right. \tag{14}$$

Theorem 2.1. *Problem (M) has a unique solution (u,λ), with continuous dependence on the data $(l_2, m_2) \in X_2' \times M_2'$, if and only if the four following conditions hold*

(i) The bilinear form a is such that for any $v \in V_2$

$$\sup_{u \in V_1, \|u\|_{X_1} \le 1} a(u,v) \ge \alpha\, \|v\|_{X_2} \tag{15}$$

where $\alpha > 0$.
(ii) For any $u \in V_1$

$$\forall\, v \in V_2 \qquad a(u,v) = 0 \Rightarrow u = 0 \tag{16}$$

(iii_1) There exists $\beta_1 > 0$ such that for any $\lambda \in M_1$

$$\sup_{v \in X_2, \|v\|_{X_2} \le 1} b_1(v,\lambda) \ge \beta_1\, \|\lambda\|_{M_1} \tag{17}$$

(iii_2) There exists $\beta_2 > 0$ such that for any $\mu \in M_2$

$$\sup_{u \in X_1, \|u\|_{X_1} \le 1} b_2(u,\mu) \ge \beta_2\, \|\mu\|_{M_2} \tag{18}$$

In addition, the conditions *(i)* , *(ii)* are equivalent to the dual conditions *(i')* , *(ii')*.
(i') The bilinear form a is such that for any $u \in V_1$,

$$\sup_{v \in V_2, \|v\|_{X_2} \le 1} a(u,v) \ge \alpha' \|u\|_{X_1} \tag{19}$$

where $\alpha' > 0$.
(ii') For any $v \in V_2$,

$$\forall\, u \in V_1 \quad a(u,v) = 0 \Rightarrow v = 0. \tag{20}$$

Another couple of equivalent conditions is *(i)+(i')*. Therefore, the problem (M) is well posed if and only if , for any $(l_1\,,\, m_1) \in X_1' \times M_1'$, so is the dual problem (M') : find $(v,\mu) \in X_2 \times M_2$ such that

$$(M') \quad \left\{ \begin{array}{ccll} a(u,v) + b_2(u,\mu) & = & < l_1\,,\, u >_{X_1',\,X_1} & \forall\, u \in X_1 \\ b_1(v,\lambda) & = & < m_1\,,\, \lambda >_{M_1',\,M_1} & \forall\, \lambda \in M_1 \end{array} \right. . \tag{21}$$

The well known particular case of Theorem 2.1 is when $X_1 = X_2 = X$, $M_1 = M_2 = M$, $b_1 = b_2$. In this case, the two null spaces V_1, V_2 are identical and coincide with the space

$$V = \{u \in X \,/\, b\,(u, \lambda) = 0 \ , \ \forall\, \lambda \in M\}. \tag{22}$$

Conditions *(i)* , *(ii)* , *(iii$_{1,2}$)* for well-posedness of problem (M) reduce to

(i) $\forall\, u \in V\,, \quad \sup_{v \in V, \|v\|_V \leq 1} \quad a\,(u, v) \geq \alpha \quad \|u\|_X$

(ii) for any $v \in V$, $\qquad a\,(u, v) = 0 \quad \forall\, u \in V \Rightarrow v = 0$

(iii) $\forall\, \lambda \in M\,, \quad \sup_{u \in X, \|u\|_X \leq 1} \quad b\,(u, \lambda) \geq \beta\, \|\lambda\|_M$ (LBB condition).

If the bilinear form a is symmetric, the conditions *(i)+(ii)* are clearly equivalent to the condition *(i)* alone. Furthermore, a sufficient condition in order to have *(i)+(ii)* is the coercivity of the form a restricted to the space $V \times V$, i.e.

$$(i') \qquad\qquad a\,(u, u) \geq \alpha\, \|u\|_X^2 \quad \forall\, u \in V\,.$$

We apply Theorem 2.1 to the two problems (P_1), (P_2) with the bilinear forms a, b_1, b_2 defined by (7),(8). The Hilbert spaces are

$$X_1 = H_{\mathrm{div}}, \;\; X_2 = (L^2)^2, \;\; M_1 = H_0^1, \;\; M_2 = L^2 \tag{23}$$

equipped with their natural norms.

Proposition 2.2. *The bilinear forms a , b_1 , b_2 fullfill the conditions (i) , (ii) , (iii_1) , (iii_2) of Theorem 2.1.*

Proof. The two properties (iii_1) , (iii_2) are precisely the (LBB) conditions for the couples of spaces $(X, M) = ((L^2)^2\,,\, H_0^1)$ and $(X, M) = (H_{\mathrm{div}},\, L^2)$ which are true, [8]. Furthermore, the two null spaces V_1, V_2 defined by

$$V_1 = \{p \in H_{\mathrm{div}} \,/\, (\mathrm{div}\ p\,,\, u)_{0,\Omega} = 0 \quad \forall\, u \in L^2\} \tag{24}$$

$$V_2 = \{p \in (L^2)^2 \,/\, (p\,,\, \nabla\, u)_{0,\Omega} = 0 \quad \forall\, u \in H_0^1\} \tag{25}$$

are identical and reduce to the space V of square integrable vector fields, with null divergence in the space of distributions $\mathcal{D}'(\Omega)^2$. For $p \in V$, $a(p, p) = |p|_{0,\Omega}^2 = \|p\|_{\mathrm{div},\Omega}^2$. Therefore, the restriction of the form a to $V \times V$ is coercive, which gives the result. $\square$

We deduce from Theorem 2.1 the following result.

Corollary 2.3. *For any $f \in L^2$*

(i) there exists a unique solution $(u, p) \in H_0^1 \times H_{\mathrm{div}}$ of (P_1) such that

$$\|u\|_{1,\Omega} + \|p\|_{\mathrm{div},\Omega} \leq C_1\, |f|_{0,\Omega} \tag{26}$$

(ii) there exists a unique solution $(v, q) \in L^2 \times (L^2)^2$ solution of (P_2) such that

$$|v|_{0,\Omega} + |q|_{0,\Omega} \leq C_2 |f|_{0,\Omega}. \tag{27}$$

In addition we verify immediately that since $H_1 = H_0^1 \times H_{\mathrm{div}} \subset L^2 \times (L^2)^2 = H_2$ with dense injection, and that $B_{|H_1 \times H_1}$ is symmetric, the solutions of the two problems (P_1), (P_2) are in fact in $H_0^1 \times H_{\mathrm{div}}$ and coincide with the unique solution $(u\,,\, \nabla\, u) \in (H_0^1 \cap H^2) \times (H^1)^2$ of problem (M).

2.2. The *FVbox* scheme of lowest order

In [22], we introduced the following finite-volume scheme for the Poisson problem, called here *FVbox scheme* for simplicity: find $(u_h\,,\,p_h) \in P^1_{nc,0} \times RT^0$ such that

$$\left\{\begin{array}{cc} \langle \operatorname{div} p_h + f, 1\!\!1_K\rangle = 0 & \forall\, K \in \mathcal{T}_h \\ \langle p_h - \nabla u_h, 1\!\!1_K\rangle = 0 & \forall\, K \in \mathcal{T}_h \\ u_h = 0 & \text{on } \partial\Omega \end{array}\right. \tag{28}$$

where $P^1_{nc,0}$ is the nonconforming space of Crouzeix-Raviart with homogeneous boundary conditions, and RT^0 the div-conforming space of Raviart-Thomas of lowest order defined respectively by

$$\begin{array}{rcl} P^1_{nc,0} & = & \{v_h \,/\, \forall\, K \in \mathcal{T}_h,\ \ v_{h|K} \in P^1(K), v_h \text{ is continuous at the middle} \\ & \text{of} & \text{each edge},\ v_h = 0 \text{ at the middle of each edge on } \partial\Omega\} \end{array} \tag{29}$$

$$RT^0 = \left\{p_h \in H_{\mathrm{div}} \,/\, p_{h|K} \in RT^0\,(K) \quad \forall\, K \in \mathcal{T}_h\right\} \tag{30}$$

where $RT^0(K)$ is the 3-dimensional space

$$RT^0(K) = (P^0(K))^2 + P^0(K)\begin{bmatrix} x^1 \\ x^2 \end{bmatrix}. \tag{31}$$

This scheme is in fact a nonconforming Petrov-Galerkin approximation of the mixed formulation (P_1). Consider any regular finite element triangulation $\mathcal{T}_h$ (in the usual sense) of the domain $\Omega \subset \mathbb{R}^2$. The primal Hilbert space is $H_1 = X_1 \times M_1 = H_{\mathrm{div}} \times H^1_0$. It is approximated by the space $K^1_h = X_{1,h} \times M_{1,h}$ with

$$X_{1,h} = RT^0, \quad M_{1,h} = P^1_{nc,0} \tag{32}$$

Since $M_{1,h} = P^1_{nc,0} \not\subset M_1 = H^1_0$, this is a nonconforming approximation. We check easily that $H_{1,h} = H_1 + K_{1,h}$ is a Hilbert space, equipped with the mesh-dependent norm $\|(u,p)\|_{1,h}$ defined by

$$\|(u,p)\|_{1,h} = (\|u\|^2_h + \|p\|^2_{\mathrm{div},\Omega})^{1/2} \tag{33}$$

where the discrete energy norm on $H^1_0 + P^1_{nc,0}$ is given by

$$\|u\|_h = (\sum_{K\in\mathcal{T}_h} |\nabla u|^2_{0,K})^{1/2} \tag{34}$$

Recall that this norm is equivalent to $(|u|^2_{0,\Omega} + \|u\|^2_h)^{1/2}$ on $H^1_0 + P^1_{nc,0}$. The norm $\|(u,p)\|_{1,h}$ extends the one of H_1 to $H_{1,h}$.

Denoting by P^0 the space of the functions constant in each triangle K, the discrete test space is $K_{2,h} = X_{2,h} \times M_{2,h}$, with

$$X_{2,h} = (P^0)^2, \quad M_{2,h} = P^0 \tag{35}$$

We have $K_{2,h} \subset H_2$. By symmetry with $H_{1,h}$, we denote $H_{2,h} = H_2 + K_{2,h} = H_2$, and for $(v,q) \in H_{2,h}$, we note

$$\|(v,q)\|_{2,h} = (|v|^2_{0,\Omega} + |q|^2_{0,\Omega})^{1/2}. \tag{36}$$

The *FVbox* scheme reads now : find $(u_h\,,\,p_h) \in P^1_{nc,0} \times RT^0$ such that for any $(v_h\,,\,q_h) \in P^0 \times (P^0)^2$

$$(P_{1,h})\left\{\begin{array}{ll} (\operatorname{div} p_h + f, v_h)_{0,\Omega} = 0 & \forall v_h \in P^0 \\ \displaystyle\sum_K (p_h - \nabla u_h, q_h)_{0,K} = 0 & \forall q_h \in (P^0)^2. \end{array}\right. \tag{37}$$

The bilinear forms a_h, $b_{1,h}$, $b_{2,h}$ are defined for $u_h \in P^1_{nc,0}$, $v_h \in P^0$, $p_h \in RT^0$, $q_h \in (P^0)^2$ by

$$a_h(p_h\,,\,q_h) = (p_h\,,\,q_h)_{0,\Omega} \quad,\quad b_{1,h}\,(q_h\,,\,u_h) = -\sum_K (\nabla\, u_h\,,\,q_h)_{0,K} \quad,\quad b_{2,h}\,(p_h\,,\,v_h) = (\mathrm{div}\; p_h\,,\,v_h)_{0,\Omega}. \tag{38}$$

Defining the continuous linear form m_2 on L^2 by

$$< m_2\,,\,v >= -(f,v)_{0,\Omega} \tag{39}$$

problem $(P_{1,h})$ can be rewritten as: find $(u_h\,,\,p_h) \in P^1_{nc,0} \times RT^0$ such that

$$\left\{\begin{array}{lcl} a_h\,(p_h\,,\,q_h) + b_{1,h}\,(q_h\,,\,u_h) &=& 0 \qquad \forall\, q_h \in (P^0)^2 \\ b_{2,h}\,(p_h\,,\,v_h) &=& < m_2\,,\,v_h > \qquad \forall\, v_h \in P^0, \end{array}\right. \tag{40}$$

or equivalently

$$B_h[(u_h,p_h);(v_h,q_h)] =< L_2;(v_h,q_h) > \tag{41}$$

where the bilinear form B_h is

$$B_h[(u_h,p_h);(v_h,q_h)] = a_h\,(p_h\,,\,q_h) + b_{1,h}\,(q_h\,,\,u_h) + b_{2,h}\,(p_h\,,\,v_h) \tag{42}$$

and the linear continuous form L_2 on $L^2 \times (L^2)^2$ is

$$< L_2;(v,q) >= -(f,v)_{0,\Omega}. \tag{43}$$

Let us denote respectively by NE , NA , NA_b , NA_i the number of triangles, edges, boundary edges and internal edges. Recall that

$$\left\{\begin{array}{l} \dim X_{1,h} = NA \;,\; \dim X_{2,h} = 2NE \\ \dim M_{1,h} = NA_i \;,\; \dim M_{2,h} = NE \end{array}\right. \tag{44}$$

and that

$$3NE = NA + NA_i = 2NA - NA_b \tag{45}$$

which is equivalent to say that

$$\dim X_{1,h} + \dim M_{1,h} = \dim X_{2,h} + \dim M_{2,h}. \tag{46}$$

We apply now to this particular case the following general stability and error estimate result for a numerical method having the form (M_h). Suppose we are given four finite-dimensional spaces $X_{1,h}, M_{1,h}, X_{2,h}, M_{2,h}$ approximating the four Hilbert spaces X_1, M_1, X_2, M_2 with possibly $X_{i,h} \not\subset X_i$, $M_{i,h} \not\subset M_i$. In addition, we call $H_i = X_i \times M_i$, $K_{i,h} = X_{i,h} \times M_{i,h}$, and we suppose that the spaces $H_{i,h} = H_i + K_{i,h}$ are Hilbert spaces equipped with norms $\|.\|_{i,h}$ extending to $H_{i,h}$ the norms of H_i. The approximation of problem (M) is: find $(u_h\,,\,\lambda_h) \in X_{1,h} \times M_{1,h}$ such that for any $v_h \in X_{2,h}$, $\mu_h \in M_{2,h}$ we have

$$(M_h) \quad \left\{\begin{array}{rl} a_h\,(u_h\,,\,v_h) \;+\; b_{1,h}\,(v_h\,,\,\lambda_h) &= < l_{2,h}\,,\,v_h >_{X'_{2,h};X_{2,h}} \\ b_{2,h}\,(u_h\,,\,\mu_h) &= < m_{2,h}\,,\,\mu_h >_{M'_{2,h};M_{2,h}} \end{array}\right. \tag{47}$$

where $l_{2,h}$, $m_{2,h}$ are approximations of l_2 , m_2, and a_h , $b_{1,h}$, $b_{2,h}$ are approximations of the forms a, b_1, b_2. Symmetrically, we introduce problem $(M_h^{'})$ which is the dual problem of (M_h): find $(v_h\,,\,\mu_h) \in X_{2,h} \times M_{2,h}$ such that for any $u_h \in X_{1,h}$, $\lambda_h \in M_{1,h}$

$$(M'_h) \quad \left\{\begin{array}{rl} a_h\,(u_h\,,\,v_h) \;+\; b_{2,h}\,(u_h\,,\,\mu_h) &= < l_{1,h}\,,\,u_h >_{X'_{1,h}\,;\,X_{1,h}} \\ b_{1,h}\,(v_h\,,\,\lambda_h) &= < m_{1,h}\,,\,\lambda_h >_{M'_{1,h}\,;\,M_{1,h}} \end{array}\right. \tag{48}$$

where $l_{1,h}$, $m_{1,h}$ are approximations of l_1 , m_1. In [35], Nicolaides describes this kind of problems in the conforming case, when in addition $a_h = a$, $b_{i,h} = b_i$. In [7], Bernardi *et al.* generalize the work of Nicolaides to the conforming case but with possibly $a_h \neq a$, $b_{i,h} \neq b_i$. Here, the only difference is that we have to work in the Hilbert spaces $H_{1,h} = H_1 + K_{1,h}$, $H_{2,h} = H_2 + K_{2,h}$, with a possible nonconformity. Let us introduce the discrete null spaces $V_{1,h}$, $V_{2,h}$

$$V_{1,h} = \{u_h \in X_{1,h} \,/\, b_{2,h}\,(u_h \,,\, \mu_h) = 0 \quad \forall\, \mu_h \in M_{2,h}\} \tag{49}$$

$$V_{2,h} = \{v_h \in X_{2,h} \,/\, b_{1,h}\,(v_h \,,\, \lambda_h) = 0 \quad \forall\, \lambda_h \in M_{1,h}\} \tag{50}$$

The discrete counterpart of Theorem 2.1 is

Theorem 2.4. *Problem $(M)_h$ has a unique solution (u_h, λ_h), with continuous dependence (uniform in h) on the data $(l_{2,h}, m_{2,h}) \in X'_{2,h} \times M'_{2,h}$, if and only if the four following conditions hold*

$(i)_h$ There exists $\overline{\alpha} > 0$ independent of h such that for any $v_h \in V_{2,h}$

$$\sup_{u_h \in V_{1,h}, \|u_h\|_{X_{1,h}} \leq 1} a_h\,(u_h \,,\, v_h) \geq \overline{\alpha}\,\|v_h\|_{X_{2,h}} \tag{51}$$

$(ii)_h$ $\dim\, X_{1,h} + \dim M_{1,h} = \dim\, X_{2,h} + \dim M_{2,h}$
$(iii_1)_h$ There exists $\overline{\beta}_1 > 0$ independent of h such that

$$\sup_{v_h \in X_{2,h}, \|v_h\|_{X_{2,h}} \leq 1} b_{1,h}\,(v_h \,,\, \lambda_h) \geq \overline{\beta}_1\, \|\lambda_h\|_{M_{1,h}} \tag{52}$$

$(iii_2)_h$ There exists $\overline{\beta}_2 > 0$ independent of h such that

$$\sup_{u_h \in X_{1,h}, \|u_h\|_{X_{1,h}} \leq 1} b_{2,h}\,(u_h \,,\, \mu_h) \geq \overline{\beta}_2\, \|\mu_h\|_{M_{2,h}} \tag{53}$$

We skip the proof, since it follows the same lines as the ones in [35], [7]. We note also, as in Sect. 2.1, that conditions $(i)_h + (ii)_h$ on the form a_h are equivalent to the dual conditions $(i')_h + (ii)_h$, where $(i')_h$ is

$$(i')_h \qquad \sup_{v_h \in V_{2,h}, \|v_h\|_{X_{2,h}} \leq 1} a_h\,(u_h \,,\, v_h) \geq \overline{\alpha}'\, \|u_h\|_{X_{1,h}} \tag{54}$$

with $\overline{\alpha}'$ independent of h. Therefore, the set of conditions $(i)_h$, $(ii)_h$, $(iii_1)_h$, $(iii_2)_h$ is equivalent to the set of conditions $(i')_h$, $(ii)_h$, $(iii_1)_h$, $(iii_2)_h$ and each of these sets is equivalent to the well-posedness of problem (M_h) or of problem (M'_h) .

We need now the standard error estimate (called second Strang's Lemma), whose proof is similar to the one when only two nonconforming Hilbert spaces X_h, M_h occur, [8]. We denote

$$B_h\,[(u_h \,,\, \lambda_h)\,;\, (v_h \,,\, \mu_h)] = a_h(u_h, v_h) + b_{1,h}(v_h, \lambda_h) + b_{2,h}(u_h, \mu_h) \tag{55}$$

and

$$< L_h\,;\, (v_h \,,\, \mu_h) >=< l_{2,h}\,;\, v_h >_{X'_{2,h};X_{2,h}} + < m_{2,h}\,;\, \mu_h >_{M'_{2,h};M_{2,h}} \tag{56}$$

we have the following

Theorem 2.5. *There exists a constant $C > 0$ independant of h such that*

$$\begin{aligned} \|(u,\lambda)-(u_h,\lambda_h)\|_{1,h} \;\leq\; & C \left\{ \inf_{(\tilde u_h,\tilde\lambda_h)\in K_{1,h}} [\, \|(u,\lambda)-(\tilde u_h,\tilde\lambda_h)\|_{1,h} \,] \right. \\ & \left. + \sup_{(\tilde v_h,\tilde\mu_h)\in K_{2,h}} \frac{|B_h\,[(u,\lambda);(\tilde v_h,\tilde\mu_h)] - < L_h\,;\,(\tilde v_h,\tilde\mu_h) > |}{\|(\tilde v_h,\tilde\mu_h)\|_{2,h}} \right\}. \end{aligned}$$

Applying Theorem 2.4 and Theorem 2.5 to discrete problem $(P_{1,h})$, we obtain the following proposition, which summarizes in a concize way the result of [22].

Proposition 2.6. *(a) Problem $(P_{1,h})$ has a unique solution $(u_h\,,\,p_h) \in P^1_{nc,0} \times RT^0$ such that*

$$\|u_h\,\|_h + \|p_h\,\|_{\mathrm{div},\Omega} \leq C\,|f|_{0,\Omega} \tag{57}$$

(b) For $f \in H^1(\Omega)$, the solution (u_h, p_h) verifies the error estimate

$$\|u-u_h\|_h + \|p-p_h\|_{\mathrm{div},\Omega} \leq C\,h\,\|f\|_{1,\Omega} \tag{58}$$

where C stands for a constant independent of h.

Proof. The spaces $X_{1,h}$, $M_{1,h}$, $X_{2,h}$, $M_{2,h}$ are specified in (32) and (35). The bilinear forms a_h, $b_{1,h}$, $b_{2,h}$, B_h are given in (38), (41). The forms $l_{2,h}$, $m_{2,h}$ are given by $< l_{2,h}\,,\,q_h >= 0$, $< m_{2,h}\,,\,v_h >= -(f,v_h)_{0,\Omega}$. This gives $\|m_{2,h}\| \leq |f|_{0,\Omega}$.
(a) Acording to Theorem 2.4, we check now the conditions $(i)_h$, $(ii)_h$, $(iii_1)_h$, $(iii_2)_h$ for the forms a_h , $b_{1,h}$, $b_{2,h}$.
• $(i)_h$: The spaces $V_{1,h}$, $V_{2,h}$ are

$$V_{1,h} = \{p \in RT^0 \,/\, (\mathrm{div}\; p, v)_{0,\Omega} = 0 \qquad \forall\, v \in P^0\} \tag{59}$$

$$V_{2,h} = \{q \in (P^0)^2 \,/\, \sum_K (q\,,\,\nabla\, u)_{0,K} = 0 \qquad \forall\, u \in P^1_{nc,0}\}. \tag{60}$$

The dimensions of these spaces are

$$\dim V_{1,h} = \dim X_{1,h} - \dim M_{2,h} = NA - NE \tag{61}$$
$$\dim V_{2,h} = \dim X_{2,h} - \dim M_{1,h} = 2NE - NA_i. \tag{62}$$

Therefore, due to (45), $\dim V_{1,h} = \dim V_{2,h}$. We prove now that in fact, $V_{1,h} = V_{2,h}$. Suppose given $p \in V_{1,h}$. Then, $\mathrm{div}\; p_{|K} = 0$ for any $K \in \mathcal{T}_h$, therefore $p \in (P^0)^2$. Moreover, for any $u \in P^1_{nc,0}$,

$$\sum_K (p\,,\,\nabla\, u)_{0,K} = -\sum_K (\mathrm{div}\;\; p, u)_{0,K} + \sum_K \int_{\partial K} (p\cdot\nu)\, u\, d\sigma \tag{63}$$
$$= \sum_K \int_{\partial K} (p\cdot\nu)\, u\, d\sigma = -\sum_{a\in A} \int_a (p\cdot\nu_a)\,[u]\, d\sigma. \tag{64}$$

where $[u]$ stands for the jump of u in the direction of ν_a. (On $\partial\Omega$, we put $[u] = -u_{\mathrm{inside}}$). Since $p \in X_{1,h} \subset H_{\mathrm{div}}$, $(p\cdot\nu_a)$ is constant along each edge a. Moreover, since u is continuous at the middle of each edge, the average of $[u]$ along a is zero. Therefore (64) = 0, which gives that $p \in V_{2,h}$. This proves that $V_{1,h} \subset V_{2,h}$. By equality of the dimensions, we obtain

$$V_{1,h} = V_{2,h} = V_h. \tag{65}$$

Consequently, $\| \|_{\mathrm{div},\Omega}$ coincides on V_h with $| |_{0,\Omega}$. Therefore the condition (i_h) holds because we have for any $q \in V_h$

$$\sup_{p \in V_h\,,\,\|p\|_{\mathrm{div},\Omega} \le 1} (p,q)_{0,\Omega} = |q|_{0,\Omega}. \tag{66}$$

• $(ii)_h$: Condition $(ii)_h$ is just (46).
• $(iii_1)_h$: We have to prove that for any $u \in M_{1,h}$

$$\sup_{q \in (P^0)^2, |q|_{0,\Omega} \le 1} b_{1,h}\,(q,u) \ge \overline{\beta}_1 \, \|u\|_h \tag{67}$$

Taking $p = \dfrac{1}{|\nabla u|_{0,\Omega}} \sum_K \nabla u_{|K} \quad \mathbb{1}_K(x)$, we get

$$\sup_{q \in (P^0)^2, |q|_{0,\Omega} \le 1} b_{1,h}\,(q,u) \ge b_{1,h}\,(p,u) = \|u\|_h \tag{68}$$

which proves the result.
• $(iii_2)_h$: This is the well-known inf-sup condition of the standard mixed method of Raviart-Thomas for the couple of spaces $(v,p) \in P^0 \times RT^0$, i.e.

$$\sup_{p \in RT^0, \|p\|_{\mathrm{div},\Omega} \le 1} b_{2,h}\,(p,v) \ge \beta_2 \, |v|_{0,\Omega}. \tag{69}$$

We refer to [36] for the proof.
(b) We prove now the error estimate (58). We deduce from Theorem 2.5 (see (42), (43) for the notation), that :

$$\begin{aligned} \|u-u_h\|_h + \|p-p_h\|_{\mathrm{div},\Omega} \le \quad & C \quad \Bigg\{ \inf_{(\tilde u_h\,,\,\tilde p_h) \in P^1_{nc,0} \times RT^0} \left[\|u-\tilde u_h\|_h + \|p-\tilde p_h\|_{\mathrm{div},\Omega} \right] \\ & + \sup_{(\tilde v_h, \tilde q_h) \in P^0 \times (P^0)^2} \frac{|B_h\,[(u,p)\,;\,(\tilde v_h\,,\,\tilde q_h)] - < L_2\,;\,(\tilde v_h\,,\,\tilde q_h) > |}{\|(\tilde v_h\,;\,\tilde q_h)\|_{2,h}} \Bigg\}. \end{aligned} \tag{70}$$

Since $p = \nabla u$ and $\mathrm{div}\, p + f = 0$, we have for any $(\tilde v_h, \tilde q_h) \in P^0 \times (P^0)^2$,

$$B_h\,[(u,p)\,;\,(\tilde v_h\,,\,\tilde q_h)] - < L_2\,;\,(\tilde v_h\,,\,\tilde q_h) >= \sum_K (p - \nabla\, u\,,\,\tilde q_h)_{0,K} + (\mathrm{div}\;p\,,\,\tilde v_h)_{0,\Omega} + (f\,;\,\tilde v_h)_{0,\Omega} = 0$$

Thus, the error method vanishes. Finally, the standard estimate of the interpolation error, ([36], [8])

$$\inf_{\tilde u_h \in P^1_{nc,0}} \|u-\tilde u_h\|_h + \inf_{\tilde p_h \in RT^0} \|p-\tilde p_h\|_{\mathrm{div},\Omega} \le Ch\,\|f\|_{1,\Omega} \tag{72}$$

yields the result. □

2.3. The dual FVbox scheme

We consider now discrete problem $(P_{2,h})$, dual of $(P_{1,h})$, connected with the mixed formulation (P_2) : find $(v_h\,,\,q_h) \in P^0 \times (P^0)^2$ such that (see (48))

$$(P_{2,h}) \begin{cases} -\sum_K (\nabla\, u_h\,,\,q_h)_{0,K} = -(f,u_h)_{0,\Omega} & \forall u_h \in P^1_{nc,0} \\ (p_h\,,\,q_h)_{0,\Omega} + (\mathrm{div}\, p_h, v_h)_{0,\Omega} = 0 & \forall p_h \in RT^0 \end{cases} \tag{73}$$

$(P_{2,h})$ can be rewritten, with the help of the form B_h : find $(v_h\,,\,q_h) \in P^0 \times (P^0)^2$ such that for any $(u_h\,,\,p_h) \in P^1_{nc,0} \times RT^0$

$$B_h\,[(u_h\,,\,p_h)\,;\,(v_h\,,\,q_h)] =< L_{1,h}\,,\,(u_h\,,\,p_h) > \tag{74}$$

where the linear form $L_{1,h} \in H'_{1,h}$ is defined for $(u_h, p_h) \in H_{1,h}$ by

$$< L_{1,h}\,;\,(u_h\,,\,p_h) >=< m_{1,h}, u_h >_{M'_{1,h}\,;\,M_{1,h}} + < l_{1,h}, p_h >_{X'_{1,h}\,;\,X_{1,h}} \tag{75}$$

with $< m_{1,h}, u_h >_{M'_{1,h}\,;\,M_{1,h}} = -(f, u_h)_{0,\Omega}$ and $< l_{1,h}, u_h >_{X'_{1,h}\,;\,X_{1,h}} = 0$. Recall that $(v, q) \in L^2 \times (L^2)^2$ is the solution of the dual problem (P_2). We have $(v, q) = (u, p)$ where (u, p) is the solution of the primal problem (P_1). Symmetrically to Proposition 2.6, we have the following result

Proposition 2.7. *(a) Problem $(P_{2,h})$ has a unique solution $(v_h\,,\,q_h) \in P^0 \times (P^0)^2$ such that*

$$|v_h|_{0,\Omega} + |q_h|_{0,\Omega} \leq C\,|f|_{0,\Omega}\,. \tag{76}$$

with C independant of h.
(b) This solution satisfies the error estimate

$$|v - v_h|_{0,\Omega} + |q - q_h|_{0,\Omega} \leq Ch\,|f|_{0,\Omega}. \tag{77}$$

with C independant of h.

Proof. (a) By Theorem 2.4, a set of necessary and sufficient conditions in order for problem $(P_{2,h})$ to be well posed are conditions $(i')_h\,,\,(ii)_h\,,\,(iii_1)_h\,,\,(iii_2)_h$. It results from the remark following Theorem 2.4, that conditions $(i')_h\,,\,(ii)_h$ are equivalent to $(iii)_h\,,\,(iv)_h$, which are true by Proposition 2.6. We conclude by verifying that $\|m_{1,h}\| \leq C|f|_{0,\Omega}$.
(b) From Theorem 2.5 applied to (48), we have the error estimate

$$\begin{aligned} |v - v_h|_{0,\Omega} + |q - q_h|_{0,\Omega} \quad &\leq \quad C\,\Bigg\{ \inf_{(\tilde v_h, \tilde q_h) \in P^0 \times (P^0)^2} \left[|v - \tilde v_h|_{0,\Omega} + |q - \tilde q_h|_{0,\Omega} \right] \\ &+ \sup_{(\tilde u_h, \tilde p_h) \in P^1_{nc,0} \times RT^0} \frac{|B_h\,[(\tilde u_h\,,\,\tilde p_h)\,;\,(v\,,\,q)] - < L_{1,h}\,;\,(\tilde u_h, \tilde p_h) > |}{\|(\tilde u_h\,;\,\tilde p_h)\|_{1,h}} \Bigg\}. \end{aligned} \tag{78}$$

Estimating the consistency error, we have for any $(\tilde u_h\,,\,\tilde p_h) \in P^1_{nc,0} \times RT^0$,

$$B_h\,[(\tilde u_h\,,\,\tilde p_h)\,;\,(v, q)] - < L_{1,h}\,,\,(\tilde u_h\,,\,\tilde p_h) > = \; (\tilde p_h\,,\,q)_{0,\Omega} + (\text{div}\;\tilde p_h\,,\,v)_{0,\Omega} - \sum_K (\nabla\,\tilde u_h\,,\,q)_{0,K} + (f, \tilde u_h)_{0,\Omega}. \tag{79}$$

Because $X_{1,h} \subset H_{\text{div}}$, we have, in view of $(5)_2$,

$$(\tilde p_h\,,\,q)_{0,\Omega} + (\text{div}\;\tilde p_h\,,\,v)_{0,\Omega} = 0 \quad \forall\,\tilde p_h \in RT^0. \tag{80}$$

Therefore, since q is in fact in H_{div},

$$\begin{aligned} -\sum_K (\nabla\,\tilde u_h\,,\,q)_{0,K} + (f\,,\,\tilde u_h)_{0,\Omega} \quad &= \quad \sum_K (\tilde u_h\,,\,\text{div}\;q)_{0,K} + (f\,,\,\tilde u_h)_{0,\Omega} - \sum_K \int_{\partial K} \tilde u_h\,(q \cdot \nu)\,d\sigma \\ &= \quad -\sum_K \sum_{e \in \partial K} \int_e \tilde u_h\,(q \cdot \nu)\,d\sigma + (\tilde u_h\,,\,\text{div}\;q + f)_{0,\Omega}. \end{aligned} \tag{81}$$

The second term in (81) is zero in view of $(P_2)_1$. The first term in (81) is rewritten as (A_i is the set of the internal edges, and A_b the set of the boundary edges)

$$\sum_{a\in A_i}\int_a [\tilde{u}_h] \quad (q\cdot\nu_a)\,d\sigma - \sum_{a\in A_b}\int_a \tilde{u}_h\ (q\cdot\nu_a)\,d\sigma \tag{82}$$

By the continuity of $\tilde{u}_h$ at the middle of each edge, and since $\tilde{u}_h\,(x_a)=0$ for x_a the middle of an edge a on $\partial\Omega$, we have (recall that $\tilde{u}_{h|a}$ is affine)

$$\int_{a\in A_i}[\tilde{u}_h]\,d\sigma = 0 \quad ; \quad \int_{a\in A_b}\tilde{u}_h\,d\sigma = 0. \tag{83}$$

Therefore, denoting by $\tilde{u}_{h,i}$, $i=1,2$, the traces of $\tilde{u}_h$ on each side of the edge a, and by $\Pi_a\,\tilde{u}_h$ the common mean value of $\tilde{u}_{h,1}$, $\tilde{u}_{h,2}$ along the edge a, we obtain that (82) can be rewritten as

$$\sum_{a\in A_i}[\int_a(\tilde{u}_{h,2}-\Pi_a\,\tilde{u}_h)\,(q\cdot\nu_a)\ d\sigma - \int_a(\tilde{u}_{h,1}-\Pi_a\tilde{u}_h)(q\cdot\nu_a)\ d\sigma] - \sum_{a\in A_b}\int_a(\tilde{u}_{h,1}-\Pi_a\,\tilde{u}_h)\,(q\cdot\nu_a)\ d\sigma. \tag{84}$$

Thus, by Lemma 3 of [24], we get for each internal edge $a\in A_i$

$$|\int_{a\in A_i}[\tilde{u}_h]\,(q\cdot\nu_a)\,d\sigma| \le Ch\,\left[\,|q|_{1,K_1}\,|\tilde{u}_h|_{1,K_1} + |q|_{1,K_2}\,|\tilde{u}_h|_{1,K_2}\right] \tag{85}$$

and for each boundary edge $a\in A_b$

$$|\int_{a\in A_b}\tilde{u}_h\,(q\cdot\nu_a)\,d\sigma| \le Ch\,|q|_{1,K_1}\,|\tilde{u}_h|_{1,K_1} \tag{86}$$

where a is oriented from K_1 towards K_2, and C is a constant independent of h. We deduce finally from (85), (86) by the Cauchy-Schwarz inequality the bound

$$\left|B_h\,[(\tilde{u}_h\,,\,\tilde{p}_h)\,;\,(v,q)] - < L_{1,h}\,;\,(\tilde{u}_h\,,\,\tilde{p}_h) > \right| \le 3\,Ch\ |q|_{1,\Omega}\,\|\tilde{u}_h\|_h.$$

which gives the following estimate of the consistency error in (78)

$$\sup_{(\tilde{u}_h,\tilde{p}_h)\in P^1_{nc,0}\times RT^0}\frac{|B_h[(\tilde{u}_h,\tilde{p}_h)\,;(v,q)] - < L_{1,h}\,;(\tilde{u}_h,\tilde{p}_h) > |}{\|\tilde{u}_h,\tilde{p}_h)\|_{1,h}} \le 3\ Ch|q|_{1,\Omega}.$$

For the interpolation error, we have the two standard estimates

$$\inf_{\tilde{v}_h\in P^0}|v-\tilde{v}_h|_{0,\Omega} \le C\,h\,|v|_{1,\Omega}\,;\ \inf_{\tilde{q}_h\in(P^0)^2}|q-\tilde{q}_h|_{0,\Omega} \le C\,h\,|q|_{1,\Omega}.$$

Since $|v|_{1,\Omega} \le \|v\|_{1,\Omega} \le |f|_{0,\Omega}$, $|q|_{1,\Omega} \le |f|_{0,\Omega}$ we get the result. □

2.4. Second order error estimate

In this section, we derive an $O\,(h^2)$ error estimate in the L^2 norm for $|u-u_h|_{0,\Omega}$. Such an estimate makes use in the standard conforming finite element method of the Aubin-Nitsche argument. Recall that this argument leads to an estimate like

$$|u-u_h|_{0,\Omega} \le C\|u-u_h\|_{1,\Omega}\ \|v-v_h\|_{1,\Omega} \tag{90}$$

where v, v_h are the continuous solution and discrete approximation of an adjoint problem. Therefore a first order error estimate in the H^1 norm for u and v yields the second order convergence of $|u-u_h|_{0,\Omega}$. Here, we follow the same principle.

Theorem 2.8. *The solution $u_h \in P^1_{nc,0}$ of the FVbox scheme $(P_{1,h})$ satisfies the second order error estimate*

$$|u-u_h|_{0,\Omega} \leq C\,h^2\,\|f\|_{1,\Omega} \tag{91}$$

where C is independant of h.

Proof. We follow the same principle as in the proof of the second order error estimate for the standard nonconforming FEM method in $P^1_{nc,0}$, [8].

We start from

$$|u-u_h|_{0,\Omega} = \sup_{g\in L^2, g\neq 0} \frac{(u-u_h, g)_{0,\Omega}}{|g|_{0,\Omega}} \tag{92}$$

To each $g \in L^2(\Omega)$, corresponds the solution $(v^g\,,\,q^g) \in L^2 \times (L^2)^2$ of problem $(P_2)_g$

$$B_h\,[(u,p)\,;\,(v^g\,,\,q^g)] = -(g,u)_{0,\Omega} \qquad \forall\,(u,p) \in H^1_0 \times H_{\rm div} \tag{93}$$

which is also the unique solution in $H^1_0 \times H_{\rm div}$ of

$$B_h\,[(v^g,q^g)\,;\,(v\,,\,q)] = -(g,v)_{0,\Omega} \qquad \forall\,(v,q) \in L^2 \times (L^2)^2. \tag{94}$$

We denote by $(v^g_h\,,\,q^g_h) \in P^0 \times (P^0)^2$ the discrete solution of problem $(P_{2,h})_g$

$$B_h\,[(\tilde u,\tilde p)\,;\,(v^g_h\,,\,q^g_h)] = -(g,\tilde u)_{0,\Omega} \qquad \forall\,(\tilde u,\tilde p) \in P^1_{nc,0} \times RT^0. \tag{95}$$

In addition, recall that $(u,p) \in H^1_0 \times H_{\rm div}$ and $(u_h,p_h) \in P^1_{nc,0} \times RT^0$ are solutions of the continuous and *FVbox* problems (4), (37).

For any $g \in L^2$, we have

$$\begin{array}{rcll} -(u-u_h,g)_{0,\Omega} &=& B_h\,[(u,p)\,;\,(v^g\,,\,q^g)] - B_h\,[(u_h,p_h)\,;\,(v^g_h\,,\,q^g_h)] & \\ &=& B_h\,[(u,p)-(u_h,p_h)\,;\,(v^g\,,\,q^g)-(v^g_h\,,\,q^g_h)] & (I) \\ &+& B_h\,[(u_h,p_h)\,;\,(v^g\,,\,q^g)-(v^g_h\,,\,q^g_h)] & (II) \\ &+& B_h\,[(u,p)-(u_h,p_h)\,;\,(v^g_h\,,\,q^g_h)]. & (III) \end{array}$$

Since $(v^g_h\,,\,q^g_h) \in P^0 \times (P^0)^2 \subset L^2 \times (L^2)^2$, we have $(III) = 0$, by subtracting (37) from (4). Moreover, we deduce from the two standard error estimates (58), (77) that

$$\begin{aligned} |(I)| &\leq C\|(u,p)-(u_h,p_h)\|_{1,h}\|(v^g,q^g)-(v^g_h,q^g_h)\|_{2,h} \qquad (96)\\ &\leq Ch^2|g|_{0,\Omega}\|f\|_{1,\Omega}. \qquad (97) \end{aligned}$$

Finally, we have for (II),

$$\begin{aligned} &-(II) = B_h\,[(u,p)-(u_h,p_h)\,,\,(v^g\,,\,q^g)] + (g\,;u-u_h) \\ =\;& \sum_K (p-p_h-\nabla(u-u_h),q^g)_{0,K} + (\mathrm{div}(p-p_h),v^g)_{0,\Omega} + (g,u-u_h)_{0,\Omega}. \end{aligned}$$

Since $\sum_K (p-p_h\,,\,q^g)_{0,K} = (p-p_h,\nabla v^g)_{0,\Omega} = -(\mathrm{div}(p-p_h)\,,\,v^g)_{0,\Omega}$, we obtain

$$-(II) = -\sum_K (\nabla(u-u_h)\,,\,q^g)_{0,K} + (g\,,\,u-u_h)_{0,\Omega} \tag{98}$$

and by the Green formula on each triangle K

$$-(II) = -\sum_K \int_{\partial K} (u - u_h)(q^g.\nu)\ d\sigma = \sum_{a\in\mathcal{A}} \int_a [u - u_h](q^g.\nu_a)\ d\sigma. \tag{99}$$

Calling Π_a the averaging operator on the edge a, we have $[u - u_h] = [u - u_h] - \Pi_a([u - u_h])$, since $[u] = 0$ and $\Pi_a[u_h] = 0$. Therefore

$$\begin{aligned}
|(II)| &\leq \sum_{a\in\mathcal{A}} \int_a |\{[u - u_h] - \Pi_a([u - u_h])\}(q^g.\nu_a)\ |d\sigma \\
&\leq 3Ch\sum_K |u - u_h|_{1,K}|q^g|_{1,K} \leq 3Ch\|u - u_h\|_h|q^g|_{1,\Omega} \\
&\leq C'h^2\|f\|_{1,\Omega}|v^g|_{2,\Omega} \leq C'h^2\|f\|_{1,\Omega}|g|_{0,\Omega}
\end{aligned}$$

Finally, we get the estimate

$$\begin{aligned}
|(u - u_h, g)_{0,\Omega}| &\leq |(I)| + |(II)| + |(III)| && (100)\\
&\leq Ch^2\|f\|_{1,\Omega}|g|_{0,\Omega}. && (101)
\end{aligned}$$

Dividing (100) by $|g|_{0,\Omega}$ and taking the suppremum on $g \neq 0$, we get the desired result. □

3. Further remarks

3.1. **Comparison with the nonconforming method**

The standard nonconforming method for (3) is: find $\tilde{u}_h \in P^1_{nc,0}$ such that

$$\sum_K (\nabla\tilde{u}_h, \nabla v_h)_{0,K} = (f, v_h)_{0,\Omega} \quad \forall v_h \in P^1_{nc,0} \tag{102}$$

For (102), the two following error estimates hold, [8]

$$\|u - u_h\|_h \leq Ch|f|_{0,\Omega} \quad |u - u_h|_{0,\Omega} \leq Ch^2|f|_{0,\Omega} \tag{103}$$

Denoting by Π the orthogonal projector form L^2 onto P^0, we have that u_h in $(P_{1,h})$ is the following modification of (102).

Proposition 3.1. *The function u_h in $(P_{1,h})$ is the solution of the scheme : find $u_h \in P^1_{nc,0}$ such that*

$$\sum_K (\nabla u_h, \nabla v_h)_{0,K} = (\Pi f, v_h)_{0,\Omega} \quad \forall v_h \in P^1_{nc,0} \tag{104}$$

Proof. Taking $q_h = \nabla v_h$ in $(P_{1,h})$ in (37), and taking in account that $\operatorname{div} p_{h|K}$ is constant, we obtain

$$\begin{aligned}
\sum_K (\nabla u_h, \nabla v_h)_{0,K} &= -\sum_K (\operatorname{div} p_h, v_h)_{0,K} - \sum_a \int_a (p_h.\nu)[v_h]\ d\sigma = -\sum_K (\operatorname{div} p_h, \Pi\ v_h)_{0,K} && (105)\\
&= (f, \Pi\ v_h)_{0,\Omega} = (\Pi\ f, v_h)_{0,\Omega} && (106)
\end{aligned}$$

□

Note that we can deduce from Prop. 3.1 that $\|u - u_h\|_h \le Ch|f|_{0,\Omega}$, which is better than (58). (The bound $\|f\|_{1,\Omega}$ is due in (58) to the interpolation error for p_h). However, the second order error estimate deduced from (104) is identical to the one in Theorem 2.8.

3.2. Comparison with the mixed method

The question arises naturally about the link between the *FVbox* solution $(u_h\,,\,p_h)$ of $(P_{1,h})$ and the solution $(\overline{u}_h\,,\,\overline{p}_h)$ of the standard mixed method of Raviart-Thomas. Recall that both $p_h\,,\,\overline{p}_h$ belong to the same space, namely the space RT^0. In addition, calling $\Pi\, v_h$ the orthogonal projection onto the space P^0 of a function $v_h \in P^1_{nc,0}$, it is interesting to compare $\overline{u}_h$ and $\Pi\, u_h$. We have

Proposition 3.2. *(i) $p_h = \overline{p}_h$. In other words, p_h concides with the approximate gradient of u provided by the mixed method.*
(ii) We have

$$\overline{u}_h = \Pi\, u_h + \frac{1}{4}\sum_K (\Pi f)_K \rho_K^2 \mathbb{1}_K \tag{107}$$

where ρ_K is the gyration radius of K defined by $|K|\,\rho_K^2 = |\overrightarrow{g_K\, x}|^2_{0,K}$ and g_K is the barycentre of K.

Proof. *(i)* The mixed scheme is : find $(\overline{u}_h\,,\,\overline{p}_h) \in P^0 \times RT^0$ such that :

$$\left\{\begin{array}{ll} (\mathrm{div}\ \overline{p}_h + f, \tilde{v})_{0,\Omega} = 0 & \forall\, \tilde{v} \in P^0 \\ (\overline{p}_h\,,\,\tilde{q})_{0,\Omega} + (\overline{u}_h\,,\,\mathrm{div}\ \tilde{q})_{0,\Omega} = 0 & \forall\, \tilde{q} \in RT^0 \end{array}\right. \tag{108}$$

whereas the *FVbox* scheme is : find $(u_h\,,\,p_h) \in P^1_{nc,0} \times RT^0$ such that

$$\left\{\begin{array}{ll} (\mathrm{div}\ p_h + f\,,\,\tilde{v})_{0,\Omega} = 0 & \forall\, \tilde{v} \in P^0 \\ (p_h\,,\,q)_{0,\Omega} - \displaystyle\sum_K (\nabla\, u_h\,,\,q)_{0,K} = 0 & \forall\, q \in (P^0)^2. \end{array}\right. \tag{109}$$

Defining $\hat{p} \in RT^0$ by $\hat{p} = p_h - \overline{p}_h$, we get by subtracting $(108)_1$, from $(109)_1$,

$$(\mathrm{div}\ \hat{p}\,,\,\tilde{v})_{0,\Omega} = 0 \qquad \forall\, \tilde{v} \in P^0 \tag{110}$$

therefore $\hat{p} \in V_h$, the common null space of the bilinear forms $b_{1,h}\,,\,b_{2,h}$, (see (65)). Recall that this space is

$$\begin{aligned} V_h &= \{p_h \in RT^0 \,/\, (\mathrm{div}\ p_h\,,\,v_h)_{0,\Omega} = 0\,, \qquad \forall\, v_h \in P^0\} \\ &= \left\{q_h \in (P^0)^2 \,/\, \sum_K (q_h\,,\,\nabla\, u_h)_{0,K} = 0\,, \qquad \forall\, u_h \in P^1_{nc,0}\right\} \\ &= RT^0 \cap (P^0)^2. \end{aligned}$$

Therefore, subtracting $(108)_2$ from $(109)_2$, we get for any $q \in V_h$:

$$(\hat{p}\,,\,q)_{0,\Omega} = \sum_K (\nabla\, u_h\,,\,q)_{0,K} + (\overline{u}_h\,,\,\mathrm{div}\, q)_{0,\Omega} = 0. \tag{111}$$

Taking $q = \hat{p}$, we obtain $\hat{p} = 0$, that is $p_h = \overline{p}_h$.
(ii) For any $\tilde{q} \in RT^0$, we estimate now $(\Pi\, u_h\,,\,\mathrm{div}\ \tilde{q})$. We have

$$(\Pi\, u_h\,,\,\mathrm{div}\ \tilde{q})_{0,\Omega} = (u_h\,,\,\mathrm{div}\ \tilde{q})_{0,\Omega} = -(\nabla u_h, \tilde{q})_{0,\Omega} - \sum_{a \in \mathcal{A}_i} \int_a [u_h](\tilde{q}\cdot\nu_a) \tag{112}$$

The second term in the right hand side of (112) vanishes ($\tilde{q} \cdot \nu_a$ is constant on a). Now, denoting g_K the barycentre of the triangle K, we have the two following identities

$$\tilde{q} = \Pi\ \tilde{q} + \frac{1}{2} \sum_K (\mathrm{div}\ \tilde{q})_K\ \overrightarrow{g_K x}\ \mathbb{1}_K \tag{113}$$

$$\nabla u_h = p_h - \frac{1}{2} \sum_K (\mathrm{div}\, p_h)_K\ \overrightarrow{g_K x}\ \mathbb{1}_K \tag{114}$$

We deduce from (113), (114) that the first term in the right hand side of (112) is

$$-(\nabla u_h, \tilde{q})_{0,\Omega} = -(p_h, \tilde{q})_{0,\Omega} + \frac{1}{4} \sum_K (\mathrm{div}\, p_h)_K\ (\mathrm{div}\, \tilde{q})_K \quad |\overrightarrow{g_K x}|^2_{0,K} \tag{115}$$

Defining the gyration radius ρ_K of the triangle K by $|K|\,\rho_K^2 = |\overrightarrow{g_K x}|^2_{0,K}$, (cf [5]), we deduce from (112), (115) and from $(\mathrm{div}\, p_h)_K = -(\Pi f)_K$ that

$$(\Pi u_h + \frac{1}{4} \sum_K (\Pi f)_K \rho_K^2 \mathbb{1}_K, \mathrm{div}\, \tilde{q})_{0,\Omega} + (p_h, \tilde{q})_{0,\Omega} = 0 \qquad \forall \tilde{q} \in RT^0 \tag{116}$$

It results from (108) that

$$\overline{u}_h = \Pi u_h + \frac{1}{4} \sum_K (\Pi f)_K \rho_K^2 \mathbb{1}_K \tag{117}$$

We deduce easily from (117) the estimate $|\Pi\, u_h - \overline{u}_h|_{0,\Omega} \leq C h^2 |f|_{0,\Omega}$. □

3.3. **Comparison with the Lagrange multipliers method**

In [2], Arnold and Brezzi describe an interpretation of the mixed method of Raviart and Thomas based on the relaxation of the divergence conformity of the $RT^0\,(K)$ element by means of a Lagrange multiplier on each edge. See also [12]. Let us describe briefly the method. We need only the lowest order version in the sequel. We call $\mathcal{RT}_0$ the finite element space constructed as

$$\mathcal{RT}_0 = \{p_h(x) = \sum_K p_{h|K}(x)\ \mathbb{1}_K\,(x)\,/\,p_{h|K} \in\ RT^0\,(K)\} \tag{118}$$

We have $\dim(\mathcal{RT}_0) = 3NE$, and $RT^0 = \mathcal{RT}_0 \cap H_{\mathrm{div}}$. The space of Lagrange multipliers is the set of functions defined only on the edges A. The lowest order version is simply the functions constant on each edge $a \in A$

$$\mathcal{S}_0^0 = \{\lambda_h\,(x) = \lambda_a\,,\ x \in a\,,\ a \in A_i\} \tag{119}$$

The scheme is : find $(\overline{u}_h\,,\overline{p}_h\,,\overline{\lambda}_h) \in P^0 \times \mathcal{RT}_0 \times \mathcal{S}_0^0$ such that for any $(\tilde{v}, \tilde{q}, \tilde{\mu}) \in P^0 \times \mathcal{RT}_0 \times \mathcal{S}_0^0$

$$\left\{ \begin{array}{rcl} \displaystyle\sum_K (\mathrm{div}\ \overline{p}_h + f\,,\,\tilde{v})_{0,K} & = & 0 \\ \displaystyle(\overline{p}_h\,,\,\tilde{q})_{0,\Omega} + \sum_K \Big[(\overline{u}_h\,,\,\mathrm{div}\ \tilde{q})_{0,K} - \int_{\partial K} \overline{\lambda}_h\,(\tilde{q}\cdot\nu)\ d\sigma\Big] & = & 0 \\ \displaystyle\sum_K \int_{\partial K} \tilde{\mu}\,(\overline{p}_h \cdot \nu) & = & 0 \end{array} \right. \tag{120}$$

System (120) has a unique solution $(\overline{u}_h\,,\overline{p}_h\,,\overline{\lambda}_h) \in P^0 \times \mathcal{RT}_0 \times \mathcal{S}_0^0$. Moreover $(\overline{u}_h\,,\overline{p}_h) \in P^0 \times RT^0$ is *the* standard mixed solution of (108). We may now lift the function $(\overline{u}_h\,,\overline{\lambda}_h)$ in an approximation $\hat{u}_h$ lying in a space of higher precision than P^0. Two choices are presented in [2]. The first one is simply to take $\hat{u}_h \in P^1_{nc}$ such that

$$\Pi_a \, \hat{u}_h = \lambda_{h,a} \qquad \forall\, a \in A_i. \tag{121}$$

The second one is to take $\hat{u}_h \in N = P^1_{nc} + B^3$ where B^3 is the *"bubble"* space

$$B^3 = \{u_h \,/\, u_{h|K} \in Vect\,(\lambda_1\,(x)\,\lambda_2\,(x)\,\lambda_3\,(x))\,,\, x \in K\}. \tag{122}$$

The lifting $\hat{u}_h$ is uniquely defined by

$$\Pi_K \, \hat{u}_h = \bar{u}_h \ ; \ \Pi_a \, \hat{u}_h = \lambda_{h,a}. \tag{123}$$

Π_a and Π_K are the orthogonal projections onto the constants in $L^2(K)$ and $L^2(a)$. In fact, it can be proved, that the lowest order version of the *FVbox* scheme coincides with the lifting (121).

Proposition 3.3. *The solution $(u_h\,,\,p_h)$ of the FVbox scheme $(P_{1,h})$ coincides with the $(\hat{u}_h\,,\,\overline{p}_h)$ solution of scheme (120), where $\hat{u}_h$ is the P^1_{nc} interpolation (121).*

Proof. Recall that $p_h = \overline{p}_h$, where $\overline{p}_h \in RT^0$ is the solution of the mixed method. For any $\tilde{q} \in \mathcal{RT}_0$, $\tilde{q} \cdot \nu$ is constant along each edge $e \in \partial K$. Therefore it is sufficient to prove that $(u_h\,,\,p_h)$ verifies, for any $\tilde{q} \in \mathcal{RT}_0$

$$(p_h\,,\,\tilde{q})_{0,\Omega} + \sum_K \Big[(\overline{u}_h\,,\,\mathrm{div}\ \tilde{q})_{0,K} - \int_{\partial K} u_h\,(\tilde{q}\cdot\nu)\ d\sigma\Big] = 0. \tag{124}$$

Since $\tilde{q}(x)_K = (\Pi\tilde{q})_K + \frac{1}{2}\,(\mathrm{div}\,\tilde{q})_K\,\overrightarrow{g_K x}$, we have

$$\begin{aligned}(p_h\,,\,\tilde{q})_{0,\Omega} &= (p_h\,,\,\Pi\,\tilde{q})_{0,\Omega} + \frac{1}{2}\Big(p_h, \sum_K (\mathrm{div}\,\tilde{q})_K \quad \overrightarrow{g_K x}\,\mathbb{1}_K\Big)_{0,\Omega} \\ &= \sum_K (\nabla\, u_h\,,\,\Pi\,\tilde{q})_{0,K} + \frac{1}{2}\Big(p_h, \sum_K (\mathrm{div}\,\tilde{q})_K\ \overrightarrow{g_K x}\ \mathbb{1}_K\Big)_{0,\Omega}\end{aligned}$$

Replacing $p_{h|K}$ by its value $p_{h|K} = (\Pi\,p_h)_K - \frac{1}{2}\,(\Pi f)_K\ \overrightarrow{g_K x}$ we find

$$\begin{aligned}(p_h, \tilde{q})_{0,\Omega} &= \sum_K (\nabla\, u_h\,,\,\tilde{q})_{0,K} - \frac{1}{4}\sum_K |K|\,(\Pi f)_K\,(\mathrm{div}\,\tilde{q})_K\,\rho_K^2 \\ &= -\sum_K \Big[(u_h\,,\,\mathrm{div}\ \tilde{q})_{0,K} - \int_{\partial K} u_h\,(\tilde{q}\cdot\nu)\ d\sigma\Big] - \frac{1}{4}\sum_K |K|(\Pi f)_K\,(\mathrm{div}\,\tilde{q})_K\rho_K^2 \\ &= -\sum_K \Big[((\Pi\,u_h)_K, \mathrm{div}\ \tilde{q})_{0,K} - \int_{\partial K} u_h\,(\tilde{q}\cdot\nu)\ d\sigma\Big] - \frac{1}{4}\sum_K |K|\,(\Pi f)_K\,(\mathrm{div}\,\tilde{q})_K\rho_K^2\end{aligned}$$

Using (107) we obtain finally

$$(p_h\,,\,\tilde{q})_{0,\Omega} = -\sum_K \Big[(\overline{u}_h\,,\,\mathrm{div}\ \tilde{q})_{0,K} - \int_{\partial K} u_h\,(\tilde{q}\cdot\nu)\ d\sigma\Big] \tag{125}$$

which is the desired result. □

3.4. **Final remark**

In this paper, we have emphasized the mixed Petrov-Galerkin structure of the scheme introduced in [22]. This structure allows to deduce the numerical analysis of this scheme from the standard theory. This approach may also be applied to schemes of the same kind with different choices of the four spaces $X_{1,h}, X_{2,h}, M_{1,h}, M_{2,h}$.

Another possibility is to eliminate p_h locally in each cell, and to obtain the error estimates for u_h from the scheme (104). The error estimate for p_h is deduced afterwards, as in [22], from the local representation formula

$$p_{h,K} = \nabla u_{h,K} - |K| \, (\Pi f)_K \; P_K(x) \tag{126}$$

where $P_K(x) = \frac{1}{2|K|}\overrightarrow{g_K x}$.

Let us mention finally that the method given by Marini, [33], for computing the mixed solution $(\overline{u}_h, \overline{p}_h)$ of Raviart-Thomas (108), amounts precisely to compute the solution u_h in (104) and to express afterwards p_h by (126). The key point of the present paper is that this a posteriori interpretation is nothing but the *FVbox* scheme (37). This scheme has the advantage to give a direct access to an approximation (u_h, p_h), affine per cell, without any reference to the mixed method.

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