Computation of the acoustic wave diffracted by an immersed elastic wedge *

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Abstract

We present a numerical method to compute accurately the acoustic wave diffracted by an elastic 2D-wedge immersed in a potential fluid, in the high-frequency limit. The principle of the method is to write the solution with the help of a *spectral function* which is the Fourier transform of layer potentials along the faces of the wedge. This meromorphic function is the solution of a singular kernel integral system. The analytical and numerical properties of the diffracted wave are entirely based on the structure of this function. This function is decomposed into an explicit polar part, corresponding to the successive reflections and refractions of the incident wave against the faces, and an holomorphic one, computed by a Galerkin/collocation method. Diffraction diagrams for various angles of the wedge and various incidence angles are displayed.

1 Introduction

Consider an infinite bidimensional wedge of angle $\varphi \in]0, \pi[$, separating an elastic medium Ω_s and a fluid Ω_f whose common boundary is $\Gamma = \Gamma_1 \cup \Gamma_2$. The longitudinal and transversal velocities in the elastic solid are c_L, c_T, and the sound velocity in the fluid is c_0 . If we choose the units such that the density of the solid $\rho_s = 1$ and the velocity $c_L = 1$, then the Lame parameters λ, μ satisfy $\lambda + 2\mu = 1$. In addition, we set $\nu_0 = 1/c_0$, $\nu_T = 1/c_T$, $\nu_L = 1$. The time harmonic dimensionless problem reads

$$(1) \qquad (E+1)\, v = 0 \ \text{ in } \ \Omega_s \qquad (\Delta + \nu_0^2)\, h = 0 \ \text{ in } \ \Omega_f$$

with the boundary conditions

$$(2) \qquad (\lambda \ \mathrm{div}\, v + 2\mu\varepsilon(v))\, \vec{n} - i\rho h\, \vec{n} = i\rho h_{in}\, \vec{n} \qquad iv \cdot \vec{n} - \mathrm{grad}\, h \cdot \vec{n} = \mathrm{grad}\, h_{in} \cdot \vec{n} \ \text{ on } \ \Gamma$$

(continuity of the static pressure and of the normal velocity along Γ). The vector v is the displacement in the solid and h the velocity potential in the fluid. The incident dimensionless plane wave in the fluid is taken as

$$(3) \qquad h_{in}(x, y) = \frac{1}{2}\ e^{i\nu_0(x\cos\theta_{in} - y\sin\theta_{in})}\,.$$

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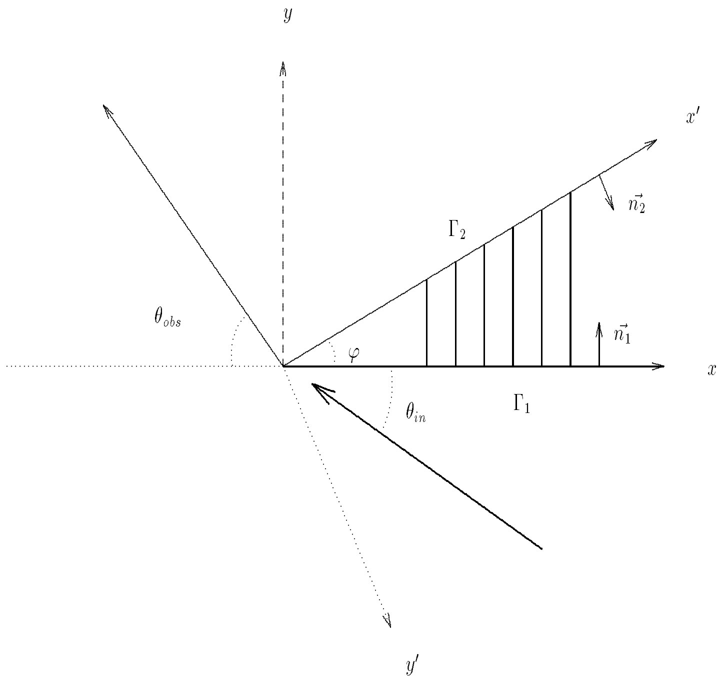

FIG. 1. *A wedge of angle φ illuminated by an incident plane wave with incidence angle θ_{in}*

Suppose that $\alpha_j, \beta_j, \gamma_j$ are potential layers supported by the faces of the wedge and define the functions $v_j,\;\; h_j$, $j = 1, 2$ by

$$v_j = -(E+1)_+^{-1}\left[\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}\otimes\delta_j\right] \qquad h_j = -(\Delta+\nu_0^2)_+^{-1}[\gamma_j\otimes\delta_j] \tag{4}$$

then the functions $v = v_1 + v_2$, $h = h_1 + h_2$ are solution of (1). In (4), the superscipt "$^{-1}$" corresponds to the inversion of the Fourier symbols of the operators $E + 1$, $\;(\Delta + \nu_0^2)$, and the subscript "$_+$" indicates that this inversion is taken in order to define properly a notion of "outgoing" solution. Moreover, δ_1 and δ_2 are the integration measures on the faces Γ_1 and Γ_2 of the wedge. In fact, an explicit one dimensional integral formula is available for (4), namely (for, say, v_1, h_1)

$$v_1(x,y) = \frac{1}{4\pi^2}\int_{\mathbb{R}} e^{ix\xi}\, L_v(\xi,y)\left[\begin{array}{c}\hat{\alpha}_1(\xi)\\ \hat{\beta}_1(\xi)\end{array}\right] d\xi\,. \tag{5}$$

$$h_1(x,y) = \frac{i}{4\pi} \int_{\mathbb{R}} e^{ix\xi} L_h(\xi,y) \hat{\gamma}_1(\xi)\, d\xi\,. \tag{6}$$

where $L_v(\xi, y)$ is a 2x2 matrix function and $L_h(\xi, y)$ is a scalar function, both known explicitly. We call *spectral function* the couple of functions (corresponding to the face $\Gamma_{1,2}$ of the wedge).

$$\Sigma_j(\xi) = \begin{bmatrix} \hat{\alpha}_j(\xi) \\ \hat{\beta}_j(\xi) \\ \hat{\gamma}_j(\xi) \end{bmatrix}, \quad j = 1, 2\,. \tag{7}$$

Applying the stationary phase theorem to (5-6) yields that the amplitude $A(\theta_{obs})$ of the diffracted wave in the fluid, in the limit $\rho^2 = |x|^2 + |y|^2 \to \infty$ in the direction of observation θ_{obs} is

$$A(\theta_{obs}) = |\hat{\gamma}_1(\nu_0 \cos\theta_{obs}) + \hat{\gamma}_2(\nu_0 \cos(\varphi - \theta_{obs}))| \tag{8}$$

(Indeed, $A(\theta_{obs})$ is the amplitude of the term in $\rho^{-1/2}$ in the asymptotics). The spectral function is therefore our basic unknown.

2 Numerical computation of the spectral function

The boundary conditions (2) are equivalent to the following system of integral equations

$$DM.\Sigma_1 + TM.\Sigma_2 = \frac{W_1}{\xi - \nu_0 \cos\theta_{in}} \quad ; \quad TM.\Sigma_1 + DM.\Sigma_2 = \frac{W_2}{\xi - \nu_0 \cos(\theta_{in} + \varphi)} \tag{9}$$

where DM and TM are integral operators with singular kernels. The first equation in (9) corresponds to Γ_1 and the second to Γ_2. The vectors $W_1, W_2 \in \mathbb{C}^3$ are given by the Fourier transforms of the incident wave h_{in}. Because Σ_1, Σ_2 are basically Fourier transforms of functions supported in $x \geq 0$, they are holomorphic in the lower complex half-plane $\Im\xi < 0$. Moreover, the system (9) permits an explicit description of the poles of Σ_j in the upper half-plane $\Im\xi > 0$. These poles are generated by a recurrence relation. The smallest is the angle of the wedge, the largest is the number of poles. Physically, they correspond to the successive reflections and refractions of the incoming wave against Γ_1, Γ_2. Substracting the polar part $y_j(\xi)$ from Σ_j, we get a remainder $X_j = \Sigma_j - y_j$, which is holomorphic in $\mathbb{C}-]-\infty, -1]$. This function is approximated by a Gallerkin/collocation function $\bar{X}_j(\xi)$. Choosing $\varphi_k(\xi) = \frac{1}{\xi + e_k}, e_k \geq 1, 1 \leq k \leq N$, as the Galerkin functions, and $b_k \geq 1$ as the collocation points, we solve the linear system whose unknowns are the components of the approximation $\bar{X}_j = \sum \alpha_j^k \varphi_k$. This system reads, $1 \leq l \leq N$)

$$\left\{ \begin{array}{lcl} DM.\bar{X}_1(b_l) + TM.\bar{X}_2(b_l) & = & \frac{W_1}{b_l - \nu_0 \cos\theta_{in}} - (DM.y_1(b_l) + TM.y_2(b_l)) \\ TM.\bar{X}_1(b_l) + DM.\bar{X}_2(b_l) & = & \frac{W_2}{b_l - \nu_0 \cos(\theta_{in} + \varphi)} - (TM.y_1(b_l) + DM.y_2(b_l)) \end{array} \right. \tag{10}$$

3 Numerical results

We display in figures 2-4 some diffraction diagrams for the couple dural/water. The plotted function is the decimal logarithm of $A(\theta_{obs})$. These curves correspond to the acoustic noise measured in debye. We observe the bifurcations corresponding to the reemission in the fluid at the longitudinal critical angles for each face $\theta_L^1 = \text{Arccos}\, \frac{1}{\nu_0} = 103°$, $\theta_L^2 = \varphi - \theta_L^1$.

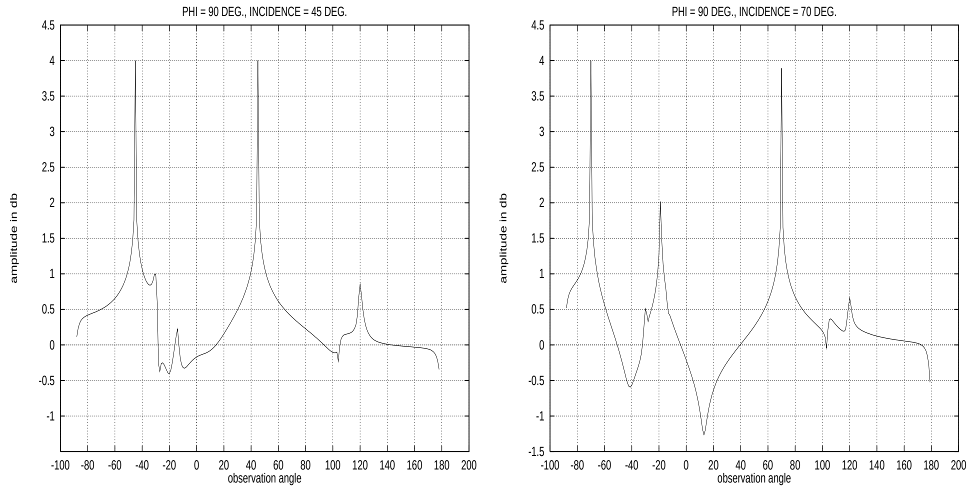

FIG. 2. *Diagramm of diffraction of a dural wedge of* $\varphi = 90°$ *immersed in water - Incident volume wave,* $\theta_{in} = 45°$ *and* $\theta_{in} = 70°$

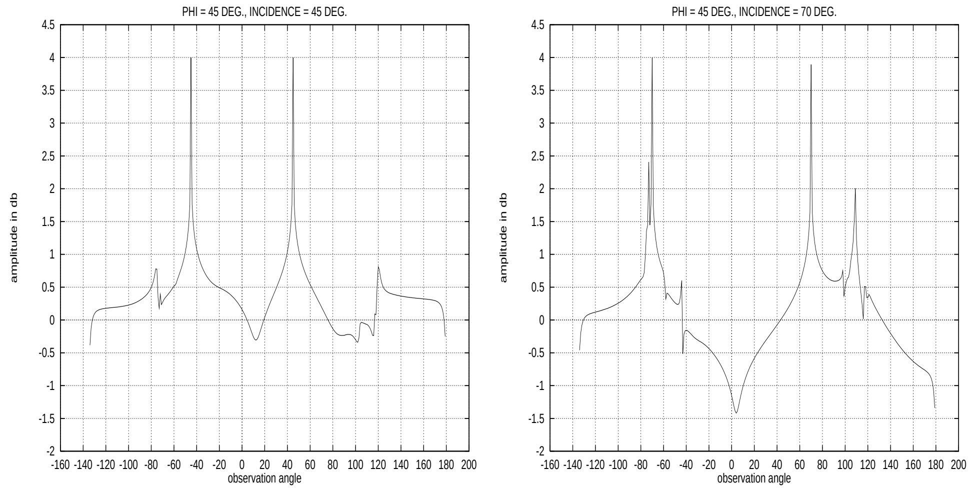

FIG. 3. *Diagramm of diffraction of a dural wedge of* $\varphi = 45°$ *immersed in water - Incident volume wave,* $\theta_{in} = 45°$ *and* $\theta_{in} = 70°$

Note that the effect of the transversal angle $\theta_T^1 = \text{Arccos}\,\frac{\nu_T}{\nu_0} = 118°$ is hidden by the one of the Rayleigh angle $\theta_R^1 = 120°$. The two large peaks correspond to the incident and reflected waves. Note also that the oscillations for the wedge $\varphi = 25°$ at $\theta_{in} = 70°$ are physical. They correspond to the excitation of the wedge by an incident wave, with an angle between the two critical angles (longitudinal and transversal).

We refer to [1] for a mathematical analysis of the problem, and for a detailed presentation of the numerical approximation. For experimental results, cf [2], [3]. For mathematical and physical results about the diffraction of a scalar wave by a wedge, cf [4], [5], [6], [7], [8], [9].

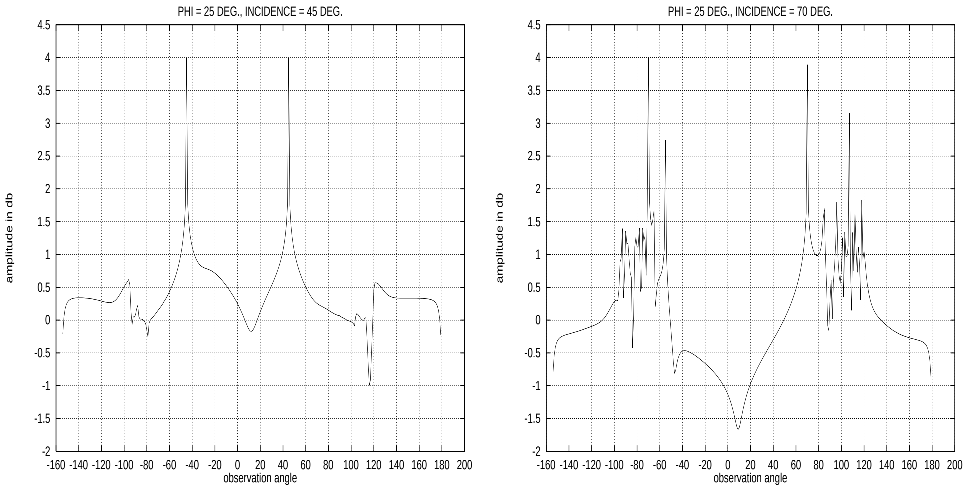

FIG. 4. *Diagramm of diffraction of a dural wedge of $\varphi = 25°$ immersed in water - Incident volume wave, $\theta_{in} = 45°$ and $\theta_{in} = 70°$*

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