RECENT DEVELOPMENTS IN THE PURE STREAMFUNCTION FORMULATION OF THE NAVIER-STOKES SYSTEM

D. FISHELOV*, M. BEN-ARTZI*, AND J.-P. CROISILLE*

Abstract. In this paper we review fourth-order approximations of the biharmonic operator in one, two and three dimensions. In addition, we describe recent developments on second and fourth order finite difference approximations of the two dimensional Navier-Stokes equations. The schemes are compact both for the biharmonic and the Laplacian operators. For the convective term the fourth order scheme invokes also a sixth order Pade approximation for the first order derivatives, using an approximation suggested by Carpenter-Gottlieb-Abarbanel in [7]. We also introduce the derivation of a pure streamfunction formulation for the Navier-Stokes equations in three dimensions.
Keywords: Navier-Stokes equations, streamfunction formulation, vorticity, numerical algorithm, compact schemes.

1. The one-dimensional Three-Point Biharmonic operator

1.1. **Design of the discrete biharmonic operator.** We consider here the one-dimensional biharmonic equation on the interval $[0, 1]$. For the simplicity of the presentation, we choose homogeneous boundary conditions. The one-dimensional biharmonic equation is

$$\begin{cases} \psi^{(4)}(x) = f(x), \quad 0 < x < 1 \\ \psi(0) = 0, \ \psi(1) = 0, \ \psi'(0) = 0, \ \psi'(1) = 0. \end{cases} \tag{1}$$

We look for a high-order compact approximation for (1). We lay out a uniform grid $0 = x_0 < x_1 < ... < x_{N-1} < x_N = 1$. Here, $x_i = ih$ for $0 \le i \le N$.

Assume that we are given data on the values of ψ and its derivative at x_{j-1}, x_j, x_{j+1}. In particular, we are given $\psi_{j-1}, \psi_j, \psi_{j+1}$, which approximate

$$\psi(x_{j-1}), \psi(x_j), \psi(x_{j+1}) \tag{2}$$

and $\psi_{x,j-1}, \psi_{x,j}, \psi_{x,j+1}$, which approximate

$$\psi'(x_{j-1}), \psi'(x_j), \psi'(x_{j+1}). \tag{3}$$

We consider a fourth order polynomial $Q(x)$,

$$Q(x) = a_0 + a_1(x - x_j) + a_2(x - x_j)^2 + a_3(x - x_j)^3 + a_4(x - x_j)^4. \tag{4}$$

For $Q(x)$ we require the five interpolation conditions

$$\begin{cases} Q(x_{j-1}) = \psi_{j-1} \quad ; \quad Q(x_j) = \psi_j \quad ; \quad Q(x_{j+1}) = \psi_{j+1} \\ Q'(x_{j-1}) = \psi_{x,j-1} \quad ; \quad Q'(x_{j+1}) = \psi_{x,j+1}. \end{cases} \tag{5}$$

As usual, we define the difference operators δ_x, δ_x^2 by

$$\delta_x \psi_j = \frac{\psi_{j+1} - \psi_{j-1}}{2h}. \tag{6}$$

$$\delta_x^2 \psi_j = \frac{\psi_{j+1} - 2\psi_j + \psi_{j-1}}{h^2}. \tag{7}$$

Date: March 3, 2010.

* Partially supported by a French-Israeli scientific cooperation grant 3-1355.

Problem (5) has a unique solution, which is given by

$$(8) \qquad \begin{cases} (a) & a_0 = \psi_j, \\ (b) & a_1 = \frac{3}{2}\delta_x\psi_j - \frac{1}{4}(\psi_{x,j+1} + \psi_{x,j-1}), \\ (c) & a_2 = \delta_x^2\psi_j - \frac{1}{2}(\delta_x\psi_x)_j, \\ (d) & a_3 = \frac{1}{h^2}(\delta_x\psi_j - \psi_{x,j}) \\ (e) & a_4 = \frac{1}{2h^2}\left((\delta_x\psi_x)_j - \delta_x^2\psi_j\right). \end{cases}$$

Therefore, it is natural to approximate $\psi^{(4)}(x_j)$ by the fourth order derivative of $Q(x)$ at point x_j. Thus,

$$(9) \qquad \psi^{(4)}(x_j) \simeq 24a_4 = \frac{12}{h^2}\left((\delta_x\psi_x)_j - \delta_x^2\psi_j\right).$$

Actually, the finite difference operator in (9) depends on the two grid functions ψ and ψ_x. If we are interested in an approximation depending only on the data ψ, then we need to construct ψ_x as a function of ψ. A natural way to derive such an approximation is to require in Equation (8)(b) that $\psi_{x,j}$ satisfy $\psi_{x,j} = a_1$ identically. This is equivalent to the following identity.

$$(10) \qquad \psi_{x,j} = \frac{3}{2}\delta_x\psi_j - \frac{1}{4}\left(\psi_{x,j+1} + \psi_{x,j-1}\right), \quad 1 \le j \le N-1$$

or equivalently

$$(11) \qquad \frac{1}{6}\psi_{x,j-1} + \frac{2}{3}\psi_{x,j} + \frac{1}{6}\psi_{x,j+1} = \delta_x\psi_j, \quad 1 \le j \le N-1.$$

If we introduce the three-point operator σ_x on grid functions by

$$(12) \qquad \sigma_x\phi_i = \frac{1}{6}\phi_{i-1} + \frac{2}{3}\phi_i + \frac{1}{6}\phi_{i+1}, \quad 1 \le i \le N-1,$$

we can rewrite (11) as

$$(13) \qquad \sigma_x(\psi_x)_i = \delta_x\psi_i, \;\; 1 \le i \le N-1.$$

Recall that

$$(14) \qquad \psi_{x,0} = \psi'_0, \quad \psi_{x,N} = \psi'_N.$$

need to be known in order to solve (13). Observe that we have the following operator equality (in the space of grid functions),

$$(15) \qquad \sigma_x = I + \frac{h^2}{6}\delta_x^2.$$

We refer to σ_x as the *Simpson operator*. Equation (11) form an implicit system of equations for $\{\psi_{x,j}\}_{j=1}^{N-1}$. This method for the approximation of the exact derivatives $\{\psi'_j\}_{j=1}^{N-1}$ is called the *Hermitian discrete gradient*. It relates the vector Ψ_x, which contains the approximate derivatives of ψ, to the vector Ψ, which contains the values of ψ at the grid points.

The approximation (9) suggests now an approximation $\delta_x^4\psi_j$ to $\psi^{(4)}(x_j)$.

$$(16) \qquad \delta_x^4\psi_j = \frac{12}{h^2}(\delta_x\psi_x - \delta_x^2\psi), \;\; 1 \le j \le N-1.$$

We refer to (16) as the *Three-Point Discrete Biharmonic* for $\psi^{(4)}$.

Using (16) and (11), the solution of (1) may be approximated by the scheme

$$(17) \qquad \begin{cases} (a) & \delta_x^4\psi_j = f(x_j) \quad 1 \le j \le N-1, \\ (b) & \frac{1}{6}\psi_{x,j-1} + \frac{2}{3}\psi_{x,j} + \frac{1}{6}\psi_{x,j+1} = \delta_x\psi_j, \quad 1 \le j \le N-1, \\ (c) & \psi_0 = 0, \; \psi_N = 0, \; \psi_{x,0} = 0, \; \psi_{x,N} = 0. \end{cases}$$

We define the discrete space

$$(18) \qquad l^2_{h,0} = \{\psi_i, 1 \le i \le N-1, \quad \psi_0 = \psi_N = 0\},$$

normed by $|\psi|_h = (h\sum_{i=1}^{N-1}\psi_i^2)^{1/2}$.

The scheme in (17) is the one-dimensional restriction of the scheme proposed by Stephenson in [13]. In the sequel, this approximation is referred to as the *one-dimensional Stephenson Scheme to the biharmonic equation.* The consistency of the scheme is given in the following proposition.

Proposition 1.1. *Suppose that $\psi(x)$ is a smooth function on $[0,1]$. Assume, in addition, that $\psi(0)=\psi(1)=0$, $\psi'(0)=\psi'(1)=0$. Let $\psi_i^* = \psi(x_i)$, $(\psi^{(4)})^*(x_i) = \psi^{(4)}(x_i)$ be the grid functions corresponding, respectively, to $\psi, \psi^{(4)}$. Then the three-point biharmonic operator δ_x^4 satisfies the following accuracy properties:*

•

$$|\sigma_x\delta_x^4\psi_i^* - \sigma_x(\psi^{(4)})^*(x_i)| \le Ch^4\|\psi^{(8)}\|_{L^\infty},\ \ 2\le i\le N-2. \tag{19}$$

• *At the near boundary point $i=1$, the fourth order accuracy of (19) drops to first order,*

$$|\sigma_x\delta_x^4\psi_1^* - \sigma_x(\psi^{(4)})^*(x_1)| \le Ch\|\psi^{(5)}\|_{L^\infty}, \tag{20}$$

with a similar estimate for $i=N-1$.

• *The error in the energy norm is given by*

$$|\delta_x^4\psi^* - (\psi^{(4)})^*|_h \le Ch^{3/2}(\|\psi^{(5)}\|_{L^\infty} + \|\psi^{(8)}\|_{L^\infty}). \tag{21}$$

In the above estimates C is a generic constant, that does not depend on ψ.

1.2. **Matrix representation of one-dimensional finite difference operators.** In this section we briefly review matrix representations of several finite difference operators, including the Three-Point Discrete Biharmonic.

1.2.1. *Centered gradient.* In the case of Dirichlet boundary conditions, the matrix representation of the operator $\psi^*\mapsto\delta_x\psi^*$ is

$$\frac{1}{2h}K\Psi, \tag{22}$$

where $\Psi = [\psi_1, \ldots\psi_{N-1}]^T$. The antisymmetric matrix $K = (K_{i,m})_{1\le i,m\le N-1}$ is

$$K = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ -1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & \ldots & -1 & 0 & 1 \\ 0 & \ldots & 0 & -1 & 0 \end{bmatrix}. \tag{23}$$

1.2.2. *Hermitian gradient.* The Hermitian gradient, defined in (17)(b), is the finite difference compact operator

$$\Psi\mapsto\Psi_x. \tag{24}$$

The vector $\Psi_x = [\psi_{x,1}, \psi_{x,2}, \cdots, \psi_{x,N-2}, \psi_{x,N-1}]$ is the solution of the linear system (11), (14). In matrix form, this system can be written as

$$\Psi_x = \frac{3}{h}P^{-1}K\Psi, \tag{25}$$

where P is the tridiagonal matrix

$$P_{i,m} = \begin{cases} 4, & m=i \\ 1, & |m-i|=1 \\ 0, & |m-i|\ge 2 \end{cases}\quad,\quad P = \begin{bmatrix} 4 & 1 & 0 & \ldots & 0 \\ 1 & 4 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & \ldots & 1 & 4 & 1 \\ 0 & \ldots & 0 & 1 & 4 \end{bmatrix}. \tag{26}$$

1.2.3. *Matrix representation of the operator* δ_x^2. The matrix representation of $\delta_x^2\Psi$ is $-\frac{1}{h^2}T\Psi$, where T is the $(N-1)\times(N-1)$ symmetric matrix

$$T = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}. \tag{27}$$

The eigenvalues of T are

$$\lambda_j = 4\sin^2\left(\frac{j\pi}{2N}\right), \quad j = 1, \cdots, N-1 \tag{28}$$

and the corresponding normalized eigenvectors are $Z^k = (Z_{1k}, \cdots, Z_{N-1,k})^T$ (with respect to the Euclidean norm in $\mathbb{R}^{N-1}$), where

$$Z_{jk} = \left(\frac{2}{N}\right)^{1/2} \sin\frac{kj\pi}{N} \quad , \quad 1 \le k, j \le N-1. \tag{29}$$

We denote the column vectors as $Z^k \in \mathbb{R}^{N-1}$ and the row vectors as $Z_j \in \mathbb{R}^{N-1}$.

The matrix $Z = (Z_{jk})_{1\le j,k\le N-1} \in \mathbb{M}_{N-1}(\mathbb{R})$ is an orthogonal positive-definite matrix. Thus,

$$Z^2 = ZZ^T = I_{N-1}. \tag{30}$$

It follows that the matrix T satisfies

$$T = Z\Lambda Z^T, \tag{31}$$

where $\Lambda = diag(\lambda_1, \cdots, \lambda_{N-1})$. The normalized vectors (with respect to $(|\cdot|_h$), which diagonalize the operator $-\delta_x^2$, are the grid functions z^k, which are defined by

$$z_{jk} = Z_{jk}/h^{1/2}. \tag{32}$$

Equivalently, they may be written as (noting that $Nh = 1$)

$$z_{jk} = \sqrt{2}\sin\frac{kj\pi}{N} \quad , \quad 1 \le k, j \le N-1. \tag{33}$$

We have

$$\begin{cases} z_{jk} = \sqrt{2}\sin(j\frac{k\pi}{N}), \quad j = 1, \cdots, N-1, \quad k = 1, ..., N-1 \\ z_{0k} = 0, \quad z_{Nk} = 0, \\ -\delta_x^2 z^k = \tilde{\lambda}_k z^k, \;\; \tilde{\lambda}_k = \frac{4}{h^2}\sin^2(\frac{k\pi}{2N}), \quad k = 1, \cdots, N-1. \end{cases} \tag{34}$$

1.2.4. *Three-Point Discrete Biharmonic operator.* The matrix form of the one-dimensional Stephenson biharmonic scheme

$$\delta_x^4\psi_i = \frac{12}{h^2}(\delta_x\psi_x - \delta_x^2\psi) \tag{35}$$

is obtained from the matrix form of operators $\psi^* \mapsto \psi_x^*$, $\psi^* \mapsto \delta_x^*\psi$ and $\psi^* \mapsto \delta_x^2\psi^*$. We obtain that the matrix representation of $\psi^* \mapsto \delta_x^4\psi^*$ is

$$S\Psi = \frac{12}{h^2}\left[\frac{3}{2h^2}KP^{-1}K + \frac{1}{h^2}T\right]\Psi = \frac{6}{h^4}\left[3KP^{-1}K + 2T\right]\Psi. \tag{36}$$

The fact that we deal with a boundary value problem, rather than a periodic one, means that $PK - KP \neq 0$. However, the commutator is non-zero only at near-boundary points. Using the precise form of this commutator, we get the following proposition. The proof can be found in [4].

Proposition 1.2. *(i) The operator* $\sigma_x\delta_x^4$ *has the matrix form*

$$PS = \frac{6}{h^4}T^2 + \frac{6}{h^4}\left[e_1(e_1 + KP^{-1}e_1)^T + e_{N-1}(e_{N-1} - KP^{-1}e_{N-1})^T\right], \tag{37}$$

where

$$e_1 = (1, 0, \cdots, 0)^T, \quad e_{N-1} = (0, \cdots, 0, 1)^T. \tag{38}$$

(ii) The symmetric positive definite operator δ_x^4 *(see (36)) has the matrix form*

$$S = \frac{6}{h^4}P^{-1}T^2 + \frac{36}{h^4}\left(V_1V_1^T + V_2V_2^T\right), \tag{39}$$

where the vectors V_1, V_2 *are*

$$\begin{cases} V_1 = (\alpha-\beta)^{1/2}P^{-1}\left(\dfrac{\sqrt{2}}{2}e_1 - \dfrac{\sqrt{2}}{2}e_{N-1}\right) \\ V_2 = (\alpha+\beta)^{1/2}P^{-1}\left(\dfrac{\sqrt{2}}{2}e_1 + \dfrac{\sqrt{2}}{2}e_{N-1}\right). \end{cases} \tag{40}$$

The constants α, β *are*

$$\begin{cases} \alpha = 2(2 - e_1^TP^{-1}e_1) \\ \beta = 2e_{N-1}^TP^{-1}e_1. \end{cases} \tag{41}$$

1.3. **Error estimate for the one-dimensional Stephenson scheme.** In [3] an error analysis based on a finite element analogy was derived. The main ingredient consists of the following observation. The bilinear pairing $<\psi,\varphi>=(\delta_x^4\psi,\varphi)_h$ defines a scalar product on the discrete space $l_{h,0}^2$ and

$$\begin{aligned} <\psi,\varphi> \quad &= \quad \sum_{i=0}^{N-1} h\frac{\psi_{x,i+1}-\psi_{x,i}}{h}\frac{\varphi_{x,i+1}-\varphi_{x,i}}{h} \\ &+ \quad \frac{12}{h^2}\sum_{i=0}^{N-1} h\left(\frac{\psi_{i+1}-\psi_i}{h} - \frac{1}{2}(\psi_{x,i}+\psi_{x,i+1})\right)\left(\frac{\varphi_{i+1}-\varphi_i}{h} - \frac{1}{2}(\varphi_{x,i}+\varphi_{x,i+1})\right). \end{aligned} \tag{42}$$

This proves the symmetry and coercivity of the bilinear form $<\psi,\varphi>$. A corollary of (42) is the following suboptimal error estimate in the energy norm $<>$ between the numerical solution $\tilde{\psi}$ and the collocated exact solution ψ^* of (1). Denoting by $e = \tilde{\psi} - \psi^*$, we have

$$<e,e>_h^{1/2} \leq Ch^{3/2}|f''|_{\infty,[0,1]}. \tag{43}$$

In order to improve (43), we use the matrix structure of δ_x^4 given in (39).

Consider again the biharmonic equation (1) and its approximation by the Stephenson scheme (17). Let ψ^* be the grid function corresponding to ψ. It satisfies

$$\delta_x^4\psi_i^* = f^*(x_i) + \mathfrak{r}_i, \qquad 1 \leq i \leq N-1, \tag{44}$$

where $\mathfrak{r}$ is by definition the truncation error. We later refer to Proposition 1.1 for estimates on $\mathfrak{r}$.

Denote by $\mathfrak{e}$ the error $\mathfrak{e} = \tilde{\psi} - \psi^*$. It satisfies

$$\begin{aligned} &\delta_x^4\mathfrak{e}_i = -\mathfrak{r}_i, \qquad 1 \leq i \leq N-1, \\ &\mathfrak{e}_0 = 0,\ \mathfrak{e}_N = 0,\ \mathfrak{e}_{x,0} = 0,\ \mathfrak{e}_{x,N} = 0. \end{aligned} \tag{45}$$

We prove the following error estimate.

Theorem 1.1. *The error* $\mathfrak{e} = \tilde{\psi} - \psi^*$ *satisfies*

$$|\mathfrak{e}|_h \leq Ch^3|log\ h|, \tag{46}$$

where C *depends only on* f.

Proof. Let $\tilde{\Psi}, \Psi^* \in \mathbb{R}^{N-1}$ be the vectors corresponding to $\tilde{\psi}, \psi^*$, respectively, and let F be the vector corresponding to f^*. We denote by $E = \tilde{\Psi} - \Psi^*$ and R the vectors corresponding to $\mathfrak{e} = \tilde{\psi} - \psi^*$ and $\mathfrak{r}$, respectively.

Using the matrix representation (39), we can write Equations (1) and (44) in the form

$$S\tilde{\Psi} = F, \tag{47}$$

and

$$S\Psi^* = F + R. \tag{48}$$

We therefore have

$$SE = -R. \tag{49}$$

Defining G as the matrix

(50) $$G = I + 6T^{-1}P^{1/2}(V_1V_1^T + V_2V_2^T)P^{1/2}T^{-1},$$

we have, in view of (39),

(51) $$S = \frac{6}{h^4}P^{-1/2}T\ G\ TP^{-1/2}.$$

As a first step we show that the matrix G is invertible and the spectral radius of its inverse is bounded by one. In fact, we have

(52) $$G = I + W_1W_1^T + W_2W_2^T,$$

where $W_1 = \sqrt{6}T^{-1}P^{1/2}V_1$, $W_2 = \sqrt{6}T^{-1}P^{1/2}V_2$. Since $W_1W_1^T + W_2W_2^T \geq 0$ (in the sense of comparison of symmetric matrices), all eigenvalues of G are greater than or equal to one. It follows that the eigenvalues of its inverse, G^{-1}, are contained in $(0,1]$ and the matrix norm of G^{-1} satisfies

(53) $$|G^{-1}| = \rho(G^{-1}) \leq 1.$$

It follows from (51) and the positivity of T, P that S is invertible and that

(54) $$\begin{aligned} E &\overset{(49)}{=} -S^{-1}R \overset{(51)}{=} -\tfrac{h^2}{\sqrt{6}}P^{1/2}T^{-1}G^{-1}\tfrac{h^2}{\sqrt{6}}T^{-1}P^{-1/2}PR \\ &= -WG^{-1}HPR, \end{aligned}$$

where

(55) $$W = \frac{h^2}{\sqrt{6}}P^{1/2}T^{-1}, \qquad H = \frac{h^2}{\sqrt{6}}T^{-1}P^{-1/2}.$$

We estimate E by the following inequality.

(56) $$|E| = |S^{-1}R| \leq |W| \cdot |HPR|.$$

- **Estimate of $|W|$.** Recall that $P = 6I - T$ (see (26), (27)), and that the eigenvalues λ_j of T are given by (28). Therefore, the eigenvalues κ_j, $1 \leq j \leq N-1$ of P are given by

(57) $$\kappa_j = 6 - \lambda_j = 6 - 4\sin^2(\frac{j\pi}{2N}), \quad 1 \leq j \leq N-1.$$

By the symmetry of W, we have that $|W| = \rho(W)$ (the spectral radius of W). Using the formulas above for λ_j and κ_j, we conclude that the eigenvalues of W are (in view of Equation (55))

$$\frac{h^2}{4\sqrt{6}}\sin^{-2}(\frac{j\pi}{2N})\sqrt{6 - 4\sin^2(\frac{j\pi}{2N})}, \quad 1 \leq j \leq N-1.$$

Noting that

$$\sqrt{6 - 4\sin^2(\frac{j\pi}{2N})} \leq \sqrt{6}, \quad \sin(\frac{j\pi}{2N}) \geq \frac{2}{\pi}(\frac{j\pi}{2N}), \quad 1 \leq j \leq N-1,$$

we conclude that

(58) $$|W| \leq C,$$

where C is independent of N.

- **Estimate of the elements of HPR.** As in (31), we can diagonalize $H = \frac{h^2}{\sqrt{6}}T^{-1}P^{-1/2}$ by

$$H = Z\Lambda' Z^T,$$

where the j-th column of the matrix Z is Z^j, as defined in (29). The diagonal matrix Λ' contains the eigenvalues of H, which can be written as

$$\theta_j = \frac{h^2}{\sqrt{6}}\lambda_j^{-1}\kappa_j^{-1/2} = \frac{h^2}{4\sqrt{6}}\frac{1}{\sin^2(\frac{j\pi}{2N})\sqrt{6 - 4\ \sin^2(\frac{j\pi}{2N})}}, \quad j = 1, \cdots, N-1.$$

The element $H_{i,k}$ of the matrix H is

$$H_{i,k} = \sum_{j=1}^{N-1} Z_{i,j}\theta_j Z_{j,k},$$

for which we have an explicit expression

$$H_{i,k} = \sum_{j=1}^{N-1} \frac{h^2}{4\sqrt{6}} \frac{2}{N} \sin(\frac{ij\pi}{N}) \frac{\sin(\frac{jk\pi}{N})}{\sin^2(\frac{j\pi}{2N})\sqrt{6-4\sin^2(\frac{j\pi}{2N})}}. \tag{59}$$

We can now estimate the order of the elements of H. In particular, we are interested in the elements of the first and the last columns of H, since they multiply extreme elements of PR, which are of order $O(h)$ (see Proposition 1.1). Columns $k = 2, ..., N-2$ of H multiply elements $k = 2, ..., N-2$ of the PR, where the latter are $O(h^4)$. We consider now elements (i,k) of H for $k = 1, N-1$. It is enough to consider $k = 1$.

$$H_{i,1} = \sum_{j=1}^{N-1} \frac{h^2}{4\sqrt{6}} \frac{2}{N} \sin(\frac{ij\pi}{N}) \frac{1}{\sin^2(\frac{j\pi}{2N})\sqrt{6-4\sin^2(\frac{j\pi}{2N})}} \sin(\frac{j\pi}{N}). \tag{60}$$

We use the following inequalities

$$\sin x \geq \frac{2}{\pi}x, \qquad 0 \leq x \leq \frac{\pi}{2}, \tag{61}$$

$$|\sin x| \leq |x|, \quad \sqrt{6-4\sin^2(\frac{j\pi}{2N})} \geq \sqrt{2}. \tag{62}$$

Noting that $h = 1/N$ and using the estimate $|\sin(\frac{ij\pi}{N})| \leq 1$, we obtain

$$|H_{i,1}| \leq C\sum_{j=1}^{N-1} h^3 \frac{1}{(jh)^2}(jh) \leq Ch^2|log\ h|. \tag{63}$$

For $k = 2, ..., N-2$ we have

$$|H_{i,k}| \leq C\sum_{j=1}^{N-1} h^3 \frac{1}{(jh)^2} \leq Ch. \tag{64}$$

Therefore,

$$\begin{aligned} &|(HPR)_i| \leq \textstyle\sum_{k=1}^{N-1} |H_{i,k}| \cdot |(PR)_k| \\ &= |H_{i,1}| \cdot |(PR)_1| + \textstyle\sum_{k=2}^{N-2} |H_{i,k}| \cdot |(PR)_k| + |H_{i,N-1}| \cdot |(PR)_{N-1}| \\ &\leq C_1h^3|log\ h| + C_2(N-3)h^5 \leq Ch^3|log\ h|. \end{aligned} \tag{65}$$

Therefore,

$$|(HPR)_i| \leq Ch^3|log\ h|, \qquad 1 \leq i \leq N-1. \tag{66}$$

Conclusion of the proof of Theorem 1.1. Using (56), (58) and (66) we obtain that the vector $E = \tilde{\Psi} - \Psi^*$ satisfies

$$\begin{aligned} |E| &= |S^{-1}R| \overset{(56)}{\leq} |W| \cdot |HPR| \overset{(58),(66)}{\leq} C\sqrt{\textstyle\sum_{i=1}^{N-1}(h^3|log\ h|)^2} \\ &= Ch^{-1/2}h^3|log\ h|. \end{aligned} \tag{67}$$

Thus,

$$|\mathfrak{e}|_h \leq Ch^3|log\ h|. \tag{68}$$

This proves the almost third order error estimate result. ■

2. Approximations of the streamfunction formulation of the Navier-Stokes equation in 2D

2.1. **Discrete Biharmonic operator in two dimensions.** The biharmonic operator $\Delta^2\psi(x,y)$ is

$$\Delta^2\psi(x,y) = \partial_x^4\psi(x,y) + \partial_y^4\psi(x,y) + 2\partial_{xy}^2\psi(x,y). \tag{69}$$

In two dimensions, the discrete Stephenson biharmonic operator is defined by

$$\Delta_h^2\psi_{i,j} = \delta_x^4\psi_{i,j} + \delta_y^4\psi_{i,j} + 2\delta_x^2\delta_y^2\psi_{i,j}. \tag{70}$$

Define the discrete gradient $(\psi_x, \psi_y) \in (l_{h,0}^2)^2$ of any $\psi \in l_{h,0}^2$ by

$$\sigma_x\psi_{x,i,j} = \delta_x\psi_{i,j} \qquad \sigma_y\psi_{y,i,j} = \delta_y\psi_{i,j} \quad , \quad 1 \le i,j \le N-1, \tag{71}$$

where σ_x, σ_y are the Simpson operators (see (13), (15)),

$$\sigma_x = I + \frac{1}{6}h^2\delta_x^2, \qquad \sigma_y = I + \frac{1}{6}h^2\delta_y^2. \tag{72}$$

The one-dimensional operators $\delta_x^4\psi_{i,j}$, $\delta_y^4\psi_{i,j}$ are given as functions of ψ, ψ_x, ψ_y by (see (16))

$$\delta_x^4\psi_{i,j} = \frac{12}{h^2}\left((\delta_x\psi_x)_{i,j} - (\delta_x^2\psi)_{i,j}\right) \ ; \ \delta_y^4\psi_{i,j} = \frac{12}{h^2}\left((\delta_y\psi_y)_{i,j} - (\delta_y^2\psi)_{i,j}\right). \tag{73}$$

The consistency error in the Stephenson operator is given by

$$\Delta_h^2\psi = \Delta^2\psi + \frac{1}{6}h^2(\partial_x^2\partial_y^4\psi + \partial_x^4\partial_y^2\psi) + O(h^4). \tag{74}$$

Therefore, Δ_h^2 is a second order approximation to the biharmonic operator Δ^2. We refer to [13],[3] for a detailed derivation of the operator Δ_h^2.

2.2. **Second order scheme for the Navier-Stokes equations.** The streamfunction formulation for the two-dimensional Navier-Stokes equations is the following evolution equation for the streamfunction ψ (see [10]).

$$\begin{cases} (a) & \partial_t(\Delta\psi) + (\nabla^\perp\psi)\cdot\nabla(\Delta\psi) - \nu\Delta^2\psi = f(x,y,t) \ , \ (x,y)\in\Omega \ , \ t>0 \\ (b) & \psi = \frac{\partial\psi}{\partial n} = 0 \ , \ (x,y)\in\partial\Omega \ , \ t>0 \\ (c) & \psi(x,y,0) = \psi_0(x,y), \ (x,y)\in\Omega. \end{cases} \tag{75}$$

Here $\mathbf{u} = \nabla^\perp\psi = (-\partial_y\psi, \partial_x\psi)$ and ν is the kinematic viscosity. The convective term in (75)(a) is $C(\psi) = \nabla^\perp\psi\cdot\nabla(\Delta\psi)$, or explicitly,

$$C(\psi) = -(\partial_y\psi)(\Delta\partial_x\psi) + (\partial_x\psi)(\Delta\partial_y\psi). \tag{76}$$

Then, (75) may be written in the following form

$$\partial_t\Delta\psi + C(\psi) - \nu\Delta^2\psi = f(x,y,t). \tag{77}$$

The spatial discretization is obtained by invoking the following second order approximations.

- *The five point discrete Laplacian*

$$\Delta_h\psi_{i,j} = \delta_x^2\psi_{i,j} + \delta_y^2\psi_{i,j}. \tag{78}$$

- *The Stephenson second-order biharmonic operator* (70), which includes the Hermitian gradient (71)
- *The convective term $C(\psi)$* is approximated by

$$C_h(\psi)_{i,j} = -\psi_{y,i,j}(\Delta_h\psi_x)_{i,j} + \psi_{x,i,j}(\Delta_h\psi_y)_{i,j}. \tag{79}$$

  The expansion of $C_h(\psi)$ in Taylor series gives

$$C_h(\psi) - C(\psi) = \tfrac{h^2}{12}\left(-\partial_y\psi\partial_x(\partial_x^4\psi + \partial_y^4\psi) + \partial_x\psi\partial_y(\partial_x^4\psi + \partial_y^4\psi)\right) + O(h^4). \tag{80}$$

Therefore, $C_h(\psi)$ is second order accurate with respect to $C(\psi)$. Finally, the semi-discrete form corresponding to (75) is (see [3])

$$\frac{d}{dt}\Delta_h\psi_{i,j}(t) + C_h(\psi(t))_{i,j} - \nu\Delta_h^2\psi_{i,j}(t) = f(x_i,y_j,t), \quad 1 \le i,j \le N-1. \tag{81}$$

We stress several important properties of (81) compared to other finite-difference schemes. First, (81) is a fully centered scheme. There is no upwinding for the convective term. In addition, the scheme is compact and relies on a nine-point stencil. The important advantage here is that there is no need to enforce any local boundary condition on the vorticity. The two boundary conditions (75)(b) are imposed on the values of ψ and its first-order derivatives, following precisely the continuous formulation.

The semi-discrete form (81) is now time-discretized by an implicit-explicit algorithm as follows. An explicit *modified Euler* for the convective term and an implicit *Crank-Nicholson* scheme for the diffusive term:

$$\left\{\begin{array}{l} \left(\Delta_h - \nu\frac{\Delta t}{4}\Delta_h^2\right)\psi_{i,j}^{n+1/2} = \Delta_h\psi_{i,j}^n - \frac{\Delta t}{2}C_h(\psi_{i,j}^n) + \nu\frac{\Delta t}{4}\Delta_h^2\psi_{i,j}^n + f_{i,j}^{n+1/4} \\ \left(\Delta_h - \nu\frac{\Delta t}{2}\Delta_h^2\right)\psi_{i,j}^{n+1} = \Delta_h\psi_{i,j}^n - \frac{\Delta t}{2}C_h(\psi_{i,j}^{n+1/2}) + \nu\frac{\Delta t}{2}\Delta_h^2\psi_{i,j}^n + f_{i,j}^{n+1/2}. \end{array}\right. \tag{82}$$

Thus, at each time step we solve two systems of linear equations. We refer to [4] for a FFT solver of each of the linear sets of equations above. This solver uses $O(NlogN)$ operations per time-step, where N is the number of points in each of the spatial directions (x or y).

Finally, we state the following convergence result for the full Navier-Stokes equation, which was proved in [3].

Theorem 2.1. *Let $T > 0$. Then, there exist constants $C, h_0 > 0$, depending possibly on T, ν and the exact solution ψ, such that, for all $0 \le t \le T$,*

$$\left(|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\right)^{1/2} \le Ch^{3/2} \quad , \quad 0 < h \le h_0 \tag{83}$$

Observe that in practice, the numerical results give usually at least second order accuracy (see [6]). Note that, as was shown in [3], the truncation error for the second-order scheme is $O(h^2)$ at interior points and $O(h)$ at near boundary points. Thus, the truncation error in l_2 is $O(h^{3/2})$ in the non-periodic case. However, in the periodic case the truncation error is $O(h^2)$. Thus, the overall accuracy is $O(h^2)$. For a fourth-order scheme, the truncation error for a periodic problem is $O(h^4)$, therefore fourth-order accuracy is recovered. We remark that in the paper by E and Liu [8] fourth-order accuracy is obtained using fourth-order approximations of vorticity boundary conditions. In the streamfunction formulation we avoid the use of vorticity boundary conditions. In particular, this allows us to extend the method to irregular domains [2].

2.3. **Fourth order scheme for the Navier-Stokes equations.** We outline the fourth order pure streamfunction scheme, presented in [5] for the equation (75).

The fourth order discrete Laplacian $\tilde{\Delta}_h\psi$ and biharmonic $\tilde{\Delta}_h^2\psi$ operators introduced in [5] are perturbations of the second order operators $\Delta_h\psi$ and $\Delta_h^2\psi$. They are designed as follows. The fourth order Laplacian is

$$\tilde{\Delta}_h\psi = 2\Delta_h\psi - (\delta_x\psi_x + \delta_y\psi_y). \tag{84}$$

Here, ψ_x, ψ_y are the fourth-order Hermitian approximations to ∂_x, ∂_y as in (71).

We note that the precise fourth-order truncation error is

$$\tilde{\Delta}_h\psi - \Delta\psi = \frac{1}{360}h^4(\partial_x^6\psi + \partial_y^6\psi) + O(h^6). \tag{85}$$

The fourth-order approximation to the biharmonic operator $\Delta^2\psi$ is

$$\tilde{\Delta}_h^2\psi = \delta_x^4\psi + \delta_y^4\psi + 2\delta_x^2\delta_y^2\psi - \frac{h^2}{6}(\delta_x^4\delta_y^2\psi + \delta_y^4\delta_x^2\psi) = \Delta^2\psi + O(h^4), \tag{86}$$

where δ_x^4 and δ_y^4 are given in (73).

The associated truncation error in (86) is

$$\tilde{\Delta}_h^2\psi - \Delta^2\psi = -h^4\left(\frac{1}{720}(\partial_x^8\psi + \partial_y^8\psi) + \frac{1}{72}\partial_x^4\partial_y^4\psi - \frac{1}{180}(\partial_x^2\partial_y^6\psi + \partial_x^6\partial_y^2\psi)\right) + O(h^6). \tag{87}$$

Recall the definition of the convective term (see (76))

$$C(\psi) = -\partial_y\psi\Delta(\partial_x\psi) + \partial_x\psi\Delta(\partial_y\psi). \tag{88}$$

Consider the term

$$\Delta(\partial_x\psi) = \partial_x^3\psi + \partial_x\partial_y^2\psi. \tag{89}$$

The mixed derivative $\partial_x\partial_y^2\psi$ may be approximated to fourth-order accuracy by $\tilde{\psi}_{yyx}$, where

$$\tilde{\psi}_{yyx} = \delta_y^2\partial_x\psi + \delta_x\delta_y^2\psi - \delta_x\delta_y\partial_y\psi. \tag{90}$$

The pure third order derivative $\partial_x^3\psi$ is approximated to fourth-order accuracy by $\tilde{\psi}_{xxx}$, where

$$(91)\qquad \begin{aligned}(\tilde{\psi}_{xxx})_{i,j} &= \frac{3}{2h^2}\left(10\delta_x\psi_{i,j} - [(\partial_x\psi)_{i+1,j} + 8(\partial_x\psi)_{i,j} + (\partial_x\psi)_{i-1,j}]\right)\\ &= \frac{3}{2h^2}\left(10\delta_x\psi - h^2\delta_x^2\partial_x\psi - 10\partial_x\psi\right)_{i,j}.\end{aligned}$$

If $\partial_x\psi$ and $\partial_y\psi$ are replaced with sixth-order accurate approximations, then (91) is a fourth-order approximation for $\partial_x^3\psi$.

Thus, a fourth-order approximation for the convective term is (see [5])

$$(92)\qquad \begin{aligned}\tilde{C}_h(\psi) &= -\psi_y\Big(\Delta_h\partial_x\psi + \tfrac{5}{2}\big(6\tfrac{\delta_x\psi-\partial_x\psi}{h^2} - \delta_x^2\partial_x\psi\big) + \delta_x\delta_y^2\psi - \delta_x\delta_y\partial_y\psi\Big)\\ &+ \psi_x\Big(\Delta_h\partial_y\psi + \tfrac{5}{2}\big(6\tfrac{\delta_y\psi-\partial_y\psi}{h^2} - \delta_y^2\partial_y\psi\big) + \delta_y\delta_x^2\psi - \delta_y\delta_x\partial_x\psi\Big)\\ &= C(\psi) + O(h^4).\end{aligned}$$

In order to retain fourth order accuracy in (92), when replacing (∂_x, ∂_y) by approximate derivatives, we have to provide a sixth order approximation for such derivatives. We denote the approximate derivatives by $\tilde{\psi}_x$ and $\tilde{\psi}_y$. Here we use a Pade relation as given in [7]. It has the following form.

$$(93)\qquad \frac{1}{3}(\tilde{\psi}_x)_{i+1,j} + (\tilde{\psi}_x)_{i,j} + \frac{1}{3}(\tilde{\psi}_x)_{i-1,j} = \frac{14}{9}\frac{\psi_{i+1,j}-\psi_{i-1,j}}{2h} + \frac{1}{9}\frac{\psi_{i+2,j}-\psi_{i-2,j}}{4h}.$$

At near-boundary points we apply a one-sided approximation for $\partial_x\psi$ (see [7]). For $i = 1$ (a point next to the left boundary) we have

$$(94)\qquad \frac{1}{10}(\tilde{\psi}_x)_{0,j} + \frac{6}{10}(\tilde{\psi}_x)_{1,j} + \frac{3}{10}(\tilde{\psi}_x)_{i-1,j} = \frac{-10\psi_{0,j} - 9\psi_{1,j} + 18\psi_{2,j} + \psi_{3,j}}{30h}.$$

For $i = N - 1$ we have

$$(95)\qquad \frac{1}{10}(\tilde{\psi}_x)_{N,j} + \frac{6}{10}(\tilde{\psi}_x)_{N-1,j} + \frac{3}{10}(\tilde{\psi}_x)_{N-2,j} = \frac{10\psi_{N,j} + 9\psi_{N-1,j} - 18\psi_{N-2,j} - \psi_{N-3,j}}{30h}.$$

In a similar manner we approximate $\partial_y\psi$.

To summarize, a fourth order approximation of the convective term is

$$(96)\qquad \begin{aligned}\tilde{C}_h(\psi) &= -\psi_y\big(\Delta_h\tilde{\psi}_x + \frac{5}{2}\big(6\frac{\delta_x\psi - \tilde{\psi}_x}{h^2} - \delta_x^2\tilde{\psi}_x\big) + \delta_x\delta_y^2\psi - \delta_x\delta_y\tilde{\psi}_y\big)\\ &+ \psi_x\big(\Delta_h\tilde{\psi}_y + \frac{5}{2}\big(6\frac{\delta_y\psi - \tilde{\psi}_y}{h^2} - \delta_y^2\tilde{\psi}_y\big) + \delta_y\delta_x^2\psi - \delta_y\delta_x\tilde{\psi}_x\big)\\ &= C(\psi) + O(h^4),\end{aligned}$$

where ψ_x, ψ_y are the Hermitian derivatives defined in (71) and $\tilde{\psi}_x, \tilde{\psi}_y$ are the approximate derivatives defined by the Pade relation for $2 \le i \le N-2, 1 \le j \le N-1$, by

$$(97)\qquad \begin{cases}\frac{1}{3}(\tilde{\psi}_x)_{i+1,j} + (\tilde{\psi}_x)_{i,j} + \frac{1}{3}(\tilde{\psi}_x)_{i-1,j} &= \frac{14}{9}\frac{\psi_{i+1,j}-\psi_{i-1,j}}{2h} + \frac{1}{9}\frac{\psi_{i+2,j}-\psi_{i-2,j}}{4h}\\ \frac{1}{10}(\tilde{\psi}_x)_{0,j} + \frac{6}{10}(\tilde{\psi}_x)_{1,j} + \frac{3}{10}(\tilde{\psi}_x)_{2,j} &= \frac{-10\psi_{0,j} - 9\psi_{1,j} + 18\psi_{2,j} + \psi_{3,j}}{30h}\\ \frac{1}{10}(\tilde{\psi}_x)_{N,j} + \frac{6}{10}(\tilde{\psi}_x)_{N-1,j} + \frac{3}{10}(\tilde{\psi}_x)_{N-2,j} &= \frac{10\psi_{N,j} + 9\psi_{N-1,j} - 18\psi_{N-2,j} - \psi_{N-3,j}}{30h}.\end{cases}$$

Analogous expressions apply to $\tilde{\psi}_y$.

Combining all fourth-order spatial discretizations with the implicit-explicit time-stepping scheme in (82) yields the following scheme.

$$(98)\qquad \frac{(\tilde{\Delta}_h\psi_{i,j})^{n+1/2} - (\tilde{\Delta}_h\psi_{i,j})^n}{\Delta t/2} = -\tilde{C}_h\psi^{(n)} + \frac{\nu}{2}[\tilde{\Delta}_h^2\psi_{i,j}^{n+1/2} + \tilde{\Delta}_h^2\psi_{i,j}^n] + f_{i,j}^{n+1/4}$$

$$(99)\qquad \frac{(\tilde{\Delta}_h\psi_{i,j})^{n+1} - (\tilde{\Delta}_h\psi_{i,j})^n}{\Delta t} = -\tilde{C}_h\psi^{(n+1/2)} + \frac{\nu}{2}[\tilde{\Delta}_h^2\psi_{i,j}^{n+1} + \tilde{\Delta}_h^2\psi_{i,j}^n] + f_{i,j}^{n+1/2}.$$

For the application of the pure streamfunction formulation on an irregular domain see [2].

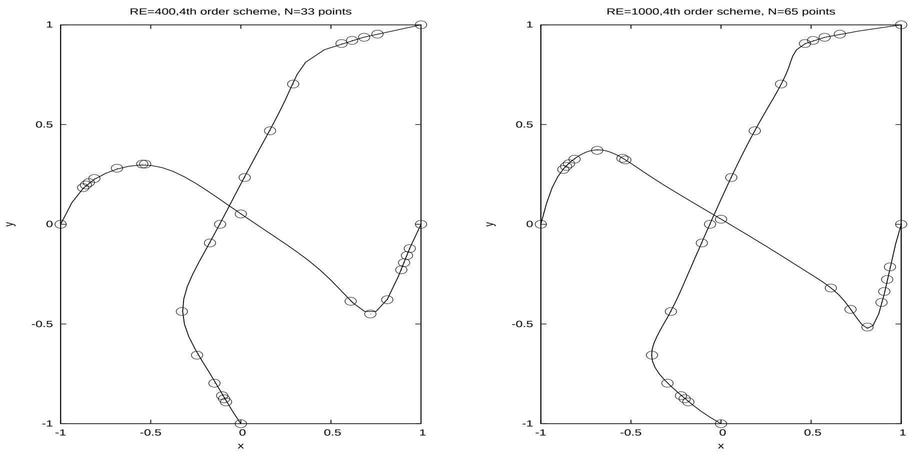

FIGURE 1. Velocity components for the driven cavity problem. Left: $Re = 400$, fourth-order scheme with $N = 33$ (solid line), [9] with $N = 129$ (circles). Right: $Re = 1000$ fourth-order scheme with $N = 65$ (solid Line), [9] with $N = 129$ (circles).

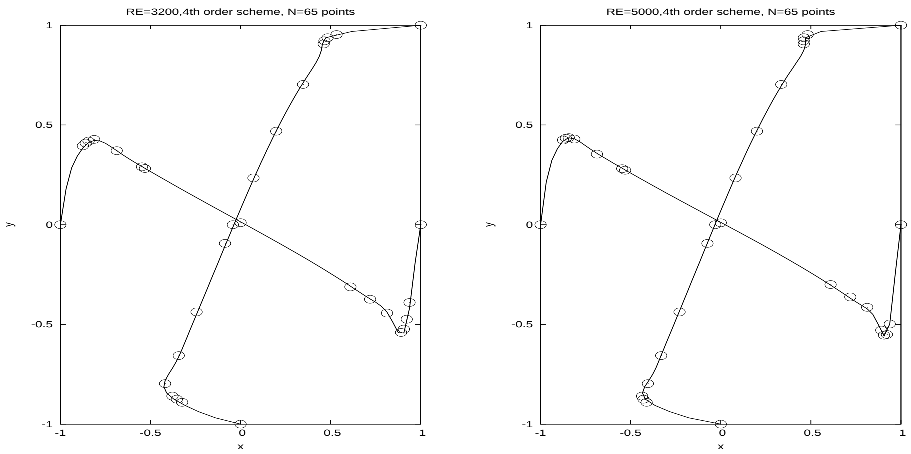

FIGURE 2. Velocity components for the driven cavity problem. Left: $Re = 3200$, fourth-order scheme with $N = 65$ (solid line), [9] with $N = 129$ (circles). Right: $Re = 5000$ fourth-order scheme with $N = 65$ (solid Line), [9] with $N = 257$ (circles).

2.4. **Numerical results for the two-dimensional Navier-Stokes equations.** We display here results for the classical driven cavity problem for Reynolds numbers 400, 1000, 3200 and 5000, using the fourth-order scheme. This problem describes a flow in a square $[0, 1] \times [0, 1]$, where on the top boundary $y = 1$ the flow is driven to the right with constant velocity $(u, v) = (1, 0)$. On all other three sides of the square - the two components of the velocity vanish. We display the results of $u(1/2, y)$ and $v(x, 1/2)$ as functions of y and x, respectively. In Fig. 1 we display the results for $Re = 400, 1000$, with $N = 33, 65$, respectively, compared with the values obtained by Ghia, Ghia and Shin [9] with $N = 129$.

In Figure 2 we display similar results for $Re = 3200, 5000$ with $N = 65$, compared to the results in [9] with $N = 129,\ N = 257$, respectively.

3. The pure streamfunction formulation in three dimensions

Let Ω be a bounded domain in R^3. The three-dimensional Navier-Stokes equations in vorticity-velocity formulation is

$$\begin{aligned} &\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) - \nu \Delta \boldsymbol{\omega} = \nabla \times \mathbf{f}, \quad \text{in} \quad \Omega \\ &\boldsymbol{\omega} = \nabla \times \mathbf{u}, \ \ \nabla \cdot \mathbf{u} = 0, \quad \text{in} \quad \Omega \\ &\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \\ &\boldsymbol{\omega}(\mathbf{x}, 0) = \boldsymbol{\omega}_0(\mathbf{x}) := \nabla \times \mathbf{u}_0, \quad \text{in} \quad \Omega. \end{aligned} \tag{100}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ and the no-slip boundary condition has been imposed. The pure streamfunction formulation for this system is obtained by introducing a streamfunction $\boldsymbol{\psi}(\mathbf{x}, t) \in R^3$, such that

$$\mathbf{u} = -\nabla \times \boldsymbol{\psi}. \tag{101}$$

This is always possible since $\nabla \cdot \mathbf{u} = \mathbf{0}$. Thus,

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \Delta \boldsymbol{\psi} - \nabla(\nabla \cdot \boldsymbol{\psi}). \tag{102}$$

Imposing a gauge condition

$$\nabla \cdot \boldsymbol{\psi} = 0, \tag{103}$$

yields

$$\omega = \Delta \boldsymbol{\psi}. \tag{104}$$

The system (100) can now be rewritten as

$$\frac{\partial \Delta \boldsymbol{\psi}}{\partial t} - \nabla \times (\Delta \boldsymbol{\psi} \times (\nabla \times \boldsymbol{\psi})) = \nu \Delta^2 \boldsymbol{\psi} + \nabla \times \mathbf{f}, \quad \text{in} \quad \Omega. \tag{105}$$

The boundary conditions $\mathbf{u} = 0$ translates to $\nabla \times \boldsymbol{\psi} = 0$ on $\partial\Omega$. We require that

$$\mathbf{n} \times \boldsymbol{\psi} = \mathbf{0}, \quad \mathbf{n} \times (\nabla \times \boldsymbol{\psi}) = \mathbf{0}, \quad \text{on} \quad \partial\Omega. \tag{106}$$

The condition $\mathbf{n} \times \boldsymbol{\psi} = \mathbf{0}$ means that $\boldsymbol{\psi}$ is parallel to $\mathbf{n}$, hence the normal component of the velocity vector is zero on the boundary. Adding the condition $\mathbf{n} \times (\nabla \times \boldsymbol{\psi}) = \mathbf{0}$ ensures that the full velocity vector vanishes on the boundary. The requirements in (106) are equivalent to four scalar conditions, namely the vanishing of the two tangential components of ψ and $\nabla \times \boldsymbol{\psi}$.

Turning now to the gauge condition $\nabla \cdot \boldsymbol{\psi} = 0$, we add the condition

$$\frac{\partial(\boldsymbol{\psi} \cdot \mathbf{n})}{\partial n} = 0, \quad \text{on} \quad \partial\Omega. \tag{107}$$

Together with the vanishing of the tangential components of $\boldsymbol{\psi}$, it implies that
$\nabla \cdot \boldsymbol{\psi} = 0$ on $\partial\Omega$.

Equations (106)-(107) consist of five scalar conditions for $\boldsymbol{\psi}$ on the boundary. We can still add one more scalar boundary condition, as the equations for the 3- component streamfunction $\boldsymbol{\psi}$ contain the fourth order biharmonic operator. The sixth scalar boundary condition that we choose to add is

$$\Delta(\nabla \cdot \boldsymbol{\psi}) = 0, \quad \text{on} \quad \partial\Omega. \tag{108}$$

We thus obtain

$$\nabla \cdot \boldsymbol{\psi} = 0, \quad \Delta(\nabla \cdot \boldsymbol{\psi}) = 0, \quad \text{on} \quad \partial\Omega. \tag{109}$$

We assume that the initial value $\boldsymbol{\psi}(\mathbf{x}, 0)$ satisfies $(\nabla \cdot \boldsymbol{\psi})(\mathbf{x}, 0) = 0$. Taking the divergence of (105) we obtain an evolution equation for $\nabla \cdot \boldsymbol{\psi}$.

$$\frac{\partial \Delta(\nabla \cdot \boldsymbol{\psi})}{\partial t} = \nu \Delta^2 (\nabla \cdot \boldsymbol{\psi}), \quad \text{in} \quad \Omega. \tag{110}$$

Equations (109)-(110) together with the assumption that $\nabla \cdot \boldsymbol{\psi} = 0$ initially ensure that $\nabla \cdot \boldsymbol{\psi} = 0$ for all $t > 0$. See also [1], [11] and [12]. Finally, we have the following three-dimensional pure streamfunction formulation

$$\begin{cases} \frac{\partial \Delta \boldsymbol{\psi}}{\partial t} - \nabla \times (\Delta \boldsymbol{\psi} \times (\nabla \times \boldsymbol{\psi})) = \nu \Delta^2 \boldsymbol{\psi} + \nabla \times \mathbf{f}, \quad \text{in} \quad \Omega \\ \mathbf{n} \times \boldsymbol{\psi} = \mathbf{0}, \frac{\partial(\boldsymbol{\psi} \cdot \mathbf{n})}{\partial n} = 0, \quad \text{on} \quad \partial\Omega \\ \mathbf{n} \times (\nabla \times \boldsymbol{\psi}) = \mathbf{0}, \quad \Delta(\nabla \cdot \boldsymbol{\psi}) = 0, \quad \text{on} \quad \partial\Omega. \end{cases} \tag{111}$$

4. The Numerical Scheme

Our numerical scheme is based on the approximation of the following equation

$$\frac{\partial \Delta \boldsymbol{\psi}}{\partial t} - ((\nabla \times \boldsymbol{\psi}) \cdot \nabla)\Delta \boldsymbol{\psi} + (\Delta \boldsymbol{\psi} \cdot \nabla)(\nabla \times \boldsymbol{\psi}) - \nu \Delta^2 \boldsymbol{\psi} = \nabla \times \mathbf{f}, \quad \text{in} \quad \Omega, \tag{112}$$

assuming that $\boldsymbol{\psi} \in H_0^2(\Omega)$. For the vector function $\boldsymbol{\psi}$ we construct a fourth-order approximation to the the biharmonic operator as follows. The pure fourth-order derivatives are approximated by $\delta_x^4, \delta_y^4, \delta_z^4$ as in (73.

The mixed term ψ_{xxyy} is approximated by

$$\tilde{\delta}_{xy}^2 \psi_{i,j,k} = 3\delta_x^2 \delta_y^2 \psi_{i,j,k} - \delta_x^2 \delta_y \psi_{y,i,j,k} - \delta_y^2 \delta_x \psi_{x,i,j,k} = \partial_x^2 \partial_y^2 \psi_{i,j,k} + O(h^4) \tag{113}$$

Similarly for ψ_{yyzz} and ψ_{zzxx}. A fourth order approximation of the biharmonic operator is then obtained as

$$\tilde{\Delta}_h^2 \psi = \delta_x^4 \psi + \delta_y^4 \psi + \delta_z^4 \psi + 2\tilde{\delta}_{xy}^2 \psi + 2\tilde{\delta}_{yz}^2 \psi + 2\tilde{\delta}_{zx}^2 \psi. \tag{114}$$

The approximate derivatives ψ_x, ψ_y and ψ_z are related to ψ via the Hermitian derivatives as in (71).

Equation (114) provides a fourth order compact operator for $\Delta^2 \psi$, which involves values of ψ, ψ_x, ψ_y and ψ_z at (i, j, k) and at its twenty six nearest neighbors. The Laplacian operator is approximated by a fourth order operator via

$$\tilde{\Delta}_h \boldsymbol{\psi} = 2\Delta_h \boldsymbol{\psi} - (\delta_x \boldsymbol{\psi}_x + \delta_y \boldsymbol{\psi}_y + \delta_z \boldsymbol{\psi}_z). \tag{115}$$

The nonlinear part in (112) consists of two terms, the convective term and the stretching term. We design a fourth-order scheme which approximates the convective term. The convective term in the three-dimensional case is

$$C(\boldsymbol{\psi}) = -((\nabla \times \boldsymbol{\psi}) \cdot \nabla)\Delta \boldsymbol{\psi} = u\Delta \partial_x \boldsymbol{\psi} + v\Delta \partial_z \boldsymbol{\psi} + w\Delta \partial_z \boldsymbol{\psi}. \tag{116}$$

Here $(u, v, w) = \mathbf{u} = -\nabla \times \boldsymbol{\psi}$ is the velocity vector, whose components contain first order derivatives of the streamfunction, and thus may be approximated to fourth-order accuracy. The terms $\Delta \partial_x \boldsymbol{\psi}, \Delta \partial_z \boldsymbol{\psi}, \Delta \partial_z \boldsymbol{\psi}$ may be approximated as in the two-dimensional case. The term $\Delta \partial_x \boldsymbol{\psi}$, for example, may be written as

$$\Delta \partial_x \boldsymbol{\psi} = \partial_x^3 \boldsymbol{\psi} + \partial_x \partial_y^2 \boldsymbol{\psi} + \partial_x \partial_z^2 \boldsymbol{\psi}. \tag{117}$$

Here, the pure and mixed type derivatives may be approximated as in the two-dimensional Navier-Stokes equations (see (91), (90)). We denote the approximation to the convective term by $\tilde{C}_h(\boldsymbol{\psi})$.

Now, we construct a fourth-order approximation to the stretching term $S = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = -(\Delta \boldsymbol{\psi} \cdot \nabla)(\nabla \times \boldsymbol{\psi})$. Note that the stretching term contains $\Delta \boldsymbol{\psi}$ and mixed second order derivatives of the streamfunction. The Laplacian of $\boldsymbol{\psi}$ may be approximated to fourth-order accuracy, as in (115). The second order mixed terms, such as $\partial_x \partial_y \boldsymbol{\psi}$, may be approximated using a Hermitian approximation of the type

$$(\sigma_x \sigma_y)(\boldsymbol{\psi}_{xy})_{i,j,k} = \delta_x \delta_y \boldsymbol{\psi}_{i,j,k}. \tag{118}$$

Hence,

$$(I + \frac{h^2}{6}\delta_x^2)(I + \frac{h^2}{6}\delta_y^2)(\boldsymbol{\psi}_{xy})_{i,j,k} = \delta_x \delta_y \boldsymbol{\psi}_{i,j,k} \quad , 1 \le i, j, k \le N-1 \tag{119}$$

is an implicit equation for $\boldsymbol{\psi}_{xy}$. We denote the approximation of the stretching term by $\tilde{S}_h(\boldsymbol{\psi})$.

Our implicit-explicit time-stepping scheme is of the Crank-Nicholson type as follows.

$$\frac{(\tilde{\Delta}_h \boldsymbol{\psi}_{i,j,k})^{n+1/2} - (\tilde{\Delta}_h \boldsymbol{\psi}_{i,j,k})^n}{\Delta t/2} = -\tilde{C}_h \boldsymbol{\psi}_{i,j,k}^{(n)} + \tilde{S}_h \boldsymbol{\psi}_{i,j,k}^{(n)} + \frac{\nu}{2}[\tilde{\Delta}_h^2 \boldsymbol{\psi}_{i,j,k}^{n+1/2} + \tilde{\Delta}_h^2 \boldsymbol{\psi}_{i,j,k}^n] \tag{120}$$

$$\frac{(\tilde{\Delta}_h \boldsymbol{\psi}_{i,j,k})^{n+1} - (\tilde{\Delta}_h \boldsymbol{\psi}_{i,j,k})^n}{\Delta t} = -\tilde{C}_h \boldsymbol{\psi}_{i,j,k}^{(n+1/2)} + \tilde{S}_h \boldsymbol{\psi}_{i,j,k}^{(n+1/2)} + \frac{\nu}{2}[\tilde{\Delta}_h^2 \boldsymbol{\psi}_{i,j}^{n+1} + \tilde{\Delta}_h^2 \boldsymbol{\psi}_{i,j,k}^n]. \tag{121}$$

Due to stability reasons we have chosen an Explicit-Implicit time stepping scheme. It is possible however to use an explicit time-stepping scheme if one can afford a small time step in order to advance the solution in time. At present, a direct solver is invoked to solve the linear set of equations (120)-(121).

Some preliminary computations with coarse grids confirm the fourth order accuracy of the scheme. We first show numerical results for the time-dependent Stokes equations

$$\frac{\partial \Delta \boldsymbol{\psi}}{\partial t} = \nu \Delta^2 \boldsymbol{\psi} + \mathbf{f}, \quad \text{in} \quad \Omega. \tag{122}$$

We have picked the exact solution $\boldsymbol{\psi}$

$$\boldsymbol{\psi}^T(\mathbf{x}, t) = -\frac{1}{4}e^{-t}\left(z^4, x^4, y^4\right) \tag{123}$$

in the cube $\Omega = (0,1)^3$. Here, $\mathbf{f}$ is chosen such that $\boldsymbol{\psi}$ in (123) satisfied (122) exactly. In the numerical results shown here we have chosen the time step Δt of order h^2 in order to retain the overall fourth-order accuracy of the scheme. In practice, if we are interested mainly in the steady state solution, a larger time step, which is independent of h, may be used. In the first table below we have picked $\Delta t = 0.1h^2$. The results for $t = 0.00625$ are given in the following Table.

	grid $5 \times 5 \times 5$	rate	grid $9 \times 9 \times 9$	rate	grid $17 \times 17 \times 17$
e	2.5460(-9)	3.82	1.8017(-10)	3.98	1.1443(-11)
e_y	7.7417(-9)	3.73	5.8037(-10)	3.96	3.7391(-11)
div ($\boldsymbol{\psi}$)	1.3409(-8)	3.74	1.0052(-9)	3.96	6.4621(-11)

Here e is the error in the l_h^2 norm, i.e.

$$e^2 = \sum_i \sum_j \sum_k (\psi_3(x_i, y_j, z_k) - \tilde{\psi}_3(x_i, y_j, z_k))^2 h^3,$$

where ψ_3 is the z component of the exact solution and $\tilde{\psi}_3$ is the z component of the approximate solution. e_y is the l_h^2 in the y derivative of ψ_3.

Below are the results with $\Delta t = h^2$ for $t = 0.0625$.

	grid $5 \times 5 \times 5$	rate	grid $9 \times 9 \times 9$	rate	grid $17 \times 17 \times 17$
e	9.6461(-7)	4.41	4.5309(-8)	4.00	2.8291(-9)
e_y	3.0293(-6)	4.33	1.5049(-7)	3.99	9.4269(-9)
div ($\boldsymbol{\psi}$)	5.2470(-6)	4.33	2.6066(-7)	4.00	1.6328(-8)

Next we show results for the Navier-Stokes Equations

$$\frac{\partial \Delta \boldsymbol{\psi}}{\partial t} - ((\nabla \times \boldsymbol{\psi}) \cdot \nabla)\Delta \boldsymbol{\psi} + (\Delta \boldsymbol{\psi} \cdot \nabla)(\nabla \times \boldsymbol{\psi}) - \nu \Delta^2 \boldsymbol{\psi} = \nabla \times \mathbf{f}, \quad \text{in} \quad \Omega \tag{124}$$

in the cube $\Omega = (0,1)^3$. Here, the source term $\mathbf{g} = \nabla \times \mathbf{f}$ is chosen such that

$$\boldsymbol{\psi}^T(\mathbf{x}, t) = -\frac{1}{4}e^{-t}\left(z^4, x^4, y^4\right)$$

is an exact solution of (124). In the following table we have picked $\Delta t = 0.1h^2$ and the results shown here are for $t = 0.00625$.

	grid $5 \times 5 \times 5$	rate	grid $9 \times 9 \times 9$	rate	grid $17 \times 17 \times 17$
e	2.4497(-9)	3.86	1.6924(-10)	4.01	1.0473(-11)
e_y	7.6486(-9)	3.75	5.6845(-10)	3.98	3.5917(-11)
div ($\boldsymbol{\psi}$)	1.2294(-8)	3.71	9.3619(-10)	3.92	6.1700(-11)

In the next table we show again results for the Navier-Stokes Equations in the cube $\Omega = (0,1)^3$, but now with $\Delta t = h^2$ for $t = 0.0625$.

	grid $5 \times 5 \times 5$	rate	grid $9 \times 9 \times 9$	rate	grid $17 \times 17 \times 17$
e	9.4418(-7)	4.46	4.2709(-8)	4.04	2.5934(-9)
e_y	2.9836(-6)	4.38	1.4334(-7)	4.03	8.7800(-9)
div ($\boldsymbol{\psi}$)	5.0471(-6)	4.40	2.3944(-7	4.02	1.4778(-8)

In the figures below we display the errors for Navier-Stokes equations in ψ_3 and $(\psi_3)_y$ at $t = 0.0625$ with $\Delta t = h^2$ and a 17^3 grid.

Acknowledgment: This paper is dedicated to the memory of Professor David Gottlieb. The first author (Dalia Fishelov) was one of his first Ph.D. students.

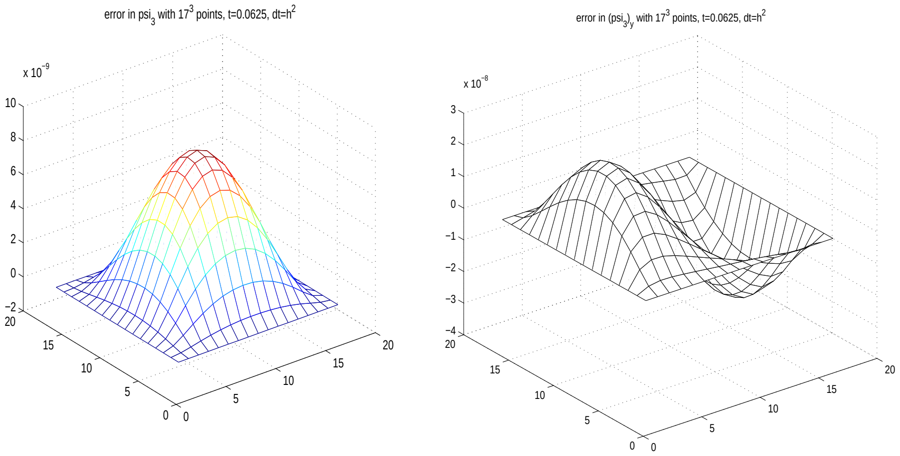

FIGURE 3. Navier-Stokes : Errors in ψ_3 and $(\psi_3)_y$ for $N = 17$, $t = 0.0625$, $dt = h^2$. The exact solution is $\boldsymbol{\psi}^T(\mathbf{x}, t) = -\frac{1}{4}e^{-t}\left(z^4, x^4, y^4\right)$.

REFERENCES

[1] M. Ben-Artzi. Planar Navier-Stokes equations, vorticity approach. In S. J. Fridlander and J. Serre, editors, *Handbook of Mathematical Fluid Dynamics*, volume II, chapter 5. Elsevier Science Publ., 2003.

[2] M. Ben-Artzi, I. Chorev, J-P. Croisille, and D. Fishelov. A compact difference scheme for the biharmonic equation in planar irregular domains. *SIAM J. Numer. Anal.*, 47(4):3087–3108, 2009.

[3] M. Ben-Artzi, J-P. Croisille, and D. Fishelov. Convergence of a compact scheme for the pure streamfunction formulation of the unsteady Navier-Stokes system. *SIAM J. Numer. Anal.*, 44,5:1997–2024, 2006.

[4] M. Ben-Artzi, J-P. Croisille, and D. Fishelov. A fast direct solver for the biharmonic problem in a rectangular grid. *SIAM J. Scient. Comp.*, 31(1):303–333, 2008.

[5] M. Ben-Artzi, J-P. Croisille, and D. Fishelov. A high order compact scheme for the pure-streamfunction formulation of the Navier-Stokes Equations. *J. Sci. Comput.*, 42(2):216–250, 2010.

[6] M. Ben-Artzi, J-P. Croisille, D. Fishelov, and S. Trachtenberg. A Pure-Compact Scheme for the Streamfunction Formulation of Navier-Stokes equations. *J. Comp. Phys.*, 205(2):640–664, 2005.

[7] M. H. Carpenter, D. Gottlieb, and S. Abarbanel. The stability of numerical boundary treatments for compact high-order schemes finite difference schemes. *J. Comput. Phys.*, 108:272–295, 1993.

[8] W. E and J.-G. Liu. Essentially compact schemes for unsteady viscous incompressible flows. *J. Comput. Phys.*, 126:122–138, 1996.

[9] U. Ghia, K. N. Ghia, and C. T. Shin. High-Re solutions for incompressible flow using the Navier-Stokes equations and a multigrid method. *J. Comput. Phys.*, 48:387–411, 1982.

[10] L. D. Landau and E. M. Lifshitz. *Fluid Mechanics, Ch.II, Sec.15.* Pergamon Press, 1959.

[11] V. Ruas and L. Quartapelle:. Uncouples finite element solutions of biharmonic problems for vector potentials. *Inter. J. Numer. Methods in Fluids*, 11:811–822, 1990.

[12] A. Rubel and G. Volpe:. Biharmonic vector stream function formulation and multigrid solutions for a three-dimensional driven-cavity stokes flow. In *AIAA Computational Fluid Dynamics Conference, 9th, Buffalo, NY, June 13-15, 1989*, pages 380–388. AIAA, 1989.

[13] J. W. Stephenson. Single cell discretizations of order two and four for biharmonic problems. *J. Comput. Phys.*, 55:65–80, 1984.

DALIA FISHELOV: AFEKA - TEL-AVIV ACADEMIC COLLEGE OF ENGINEERING, 218 BNEI-EFRAIM ST., TEL-AVIV 69107, ISRAEL
E-mail address: daliaf@post.tau.ac.il

MATANIA BEN-ARTZI: INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY, JERUSALEM 91904, ISRAEL
E-mail address: mbartzi@math.huji.ac.il

JEAN-PIERRE CROISILLE: DEPARTMENT OF MATHEMATICS, LMAM, UMR 7122, UNIVERSITY OF PAUL VERLAINE-METZ, METZ 57045, FRANCE
E-mail address: croisil@poncelet.univ-metz.fr