Keller's box-scheme for the one-dimensional stationary convection-diffusion equation *

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Abstract

The box-scheme of H.B. Keller, initially derived in [22] for the one-dimensional heat equation, is a mixed finite volume scheme for conservative equations. The basic principle of the scheme for equations like $\operatorname{div}\varphi(u, \nabla u) = f$, is to take the average onto the same mesh of the two equations of the mixed form, the conservation law $\operatorname{div} p = f$ and the constitutive law $p = \varphi(u, \nabla u)$. In this paper, we perform the numerical analysis of two Keller-like box-schemes for the one-dimensional convection-diffusion equation $cu_x - \varepsilon u_{xx} = f$. In the first one, introduced by B. Courbet in [9, 10], the numerical average of the diffusive flux is upwinded along the sign of the velocity, giving a first order accurate scheme. The second one is fourth order accurate. It is based onto the Euler-MacLaurin quadrature formula for the average of the diffusive flux. We emphasize in each case the link with the SUPG finite element method.

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1 Introduction

The box-scheme of H.B. Keller [22], is basically a mixed finite volume method, which consists in taking the average of a conservation law and of the associated constitutive law at the level of the same mesh cell. In particular, only one mesh is used as well as support of the degrees of freedom as for averaging the two equations. This design is different from the one of the "cell-centered" or "cell-vertex" finite volume methods. Recall that the cell-centered finite volume method uses numerical flux formulas at the interfaces of the mesh, involving two cells, or even more (MUSCL method). In the cell-vertex finite volume method, (also called "box-method" or "control volume method", [17, 2, 16, 4]), two different meshes are used, one as support of the degrees of freedom, and the second for averaging the equations. The price to pay for the coherence of the box-scheme formulation is a careful counting of the total degrees of freedom of the conservative and flux

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unknowns, and of the number of the test functions. In [11, 12, 13], it has been proved on the 2D Poisson problem, that several natural generalizations of the scheme of Keller are possible in multidimensions, by using FEM spaces originating from the mixed finite-element literature. The scheme results in a mixed Petrov-Galerkin scheme, using piecewise affine interpolants both for the primary unknowns and for the flux. In particular cases, one obtains a scheme strongly related to the hybridization of classical mixed finite element methods, [1]

The extension of the box-scheme to convection-dominated flows has already been addressed in [30, 29, 7] in the context of compressible aerodynamics computations. Here, we are interested in the design and numerical analysis of box-schemes for the stationary convection-diffusion equation $cu_x - \varepsilon u_{xx} = f(x)$, in the spirit of the method of B. Courbet introduced in [9]. As in the elliptic (or parabolic) case, the basic principle remains to introduce the diffusion flux $p = -\varepsilon u_x$ as an auxiliary variable. Two schemes are considered. In the first one, an upwinding of the average $\overline{p}_K$ over a "box" K (a cell), is introduced. The expected effect is to prevent the well-known spatial exponential instability at a stationary state $u(x)$, solution of $cu_x - \varepsilon u_{xx} = 0$, especially in boundary layers. However, the resulting scheme is only first order accurate in the finite-difference sense. This is the same effect as the one due to the upwinding of convective derivative in finite difference or finite element methods. The second scheme uses as a main tool the fourth order accurate Euler-MacLaurin quadrature formula on intervals, as suggested in [9]. In this case, we obtain a fourth order accurate scheme, in the finite difference sense. We emphasize the following properties of these two schemes. Firstly, they are mixed schemes with a purely local reconstruction formula for the flux. This is a well known property in the mixed finite element method for elliptic problems, [1, 24, 12], but it is less standard for convective dominated equations. Secondly, they admit a formulation as non-standard versions of the streamline-upwind-Petrov-Galerkin (SUPG) method [3, 18, 19, 20, 21, 26] with a particular design of upwind parameters. Note that the design of well-suited upwind parameters for such methods is still an active research topic, [15]. The methodology introduced here can provide interesting possibilities for designing such parameters. Finally, let us mention that there is currently a renewal of interest in the design of so-called finite difference "compact schemes", for various applications, (convection diffusion, wave equation, ...) [23, 28, 5, 6, 31], after earlier works in the '70, such as [8]. We believe that the formalism of the box-scheme of Keller can bring some clarity in the design of such schemes, allowing to work directly on unstructured meshes, instead of in the finite difference framework.

The present paper is a first attempt to derive in a systematic way high order compact schemes of mixed type in the spirit of [22, 9, 10, 12, 13], one of the applications being computing flows in porous media. The numerical simulation of such flows gives still rise to a considerable amount of works in which the mixed finite element method, the discontinuous Galerkin method, the finite volume method or the control volume method are widely used.[1]. For general references on finite volumes from a mathematical point of view, we refer to the books [14, 17, 25].

The outline of the paper is as follows. After stating the notation in Sect.2, we recall in Sect. 3 in the case of the 1D Poisson equation, the design of the box-scheme of Keller, before to perform its numerical analysis in the FEM framework. A new fourth-order box-scheme is afterwards introduced and analyzed in the same way. In Sect.4, we perform the numerical analysis of a box-scheme for the stationary convection-diffusion equation previously proposed by B. Courbet in [9, 10]. Finally, in Sect.5, we show how the fourth order scheme of Sect.3 can be extended to

[1]We refer in particular to journals such as *Water Resources Research, Journal of Contaminant Hydrology, Journal of Hydrology,...*

the convection-diffusion equation. For both schemes, the link with the SUPG method is stressed. Numerical results as well as extensions to the unstationary convection-diffusion equation or to multidimensions will be reported elsewhere.

2 Notation

In all the paper, we consider the linear stationary convection-diffusion equation with constant coefficients in the segment $I =]0\,,\,1[$. Recall that this equation is

$$\begin{cases} cu_x - \varepsilon u_{xx} = f(x), & x \in I \\ u\,(0) = 0, \quad u\,(1) = 0 \end{cases} \tag{1}$$

The velocity is $c \in \mathbb{R}$ and the diffusion coefficient is $\varepsilon > 0$. Problem (1) is well posed in $H_0^1(I)$, with data $f \in L^2(I)$. The solution is

$$u(x) = \frac{g_c(x/\varepsilon)}{g_c(1/\varepsilon)} \int_0^1 g_c((1-s)/\varepsilon) f(s) ds - \int_0^x g_c((x-s)/\varepsilon) f(s) ds \tag{2}$$

with $g_c(x) = (e^{cx} - 1)/c$ if $c \neq 0$ and $g_0(x) = x$. Let the interval I be discretized by a finite element, possibly irregular mesh, with nodes $x_1 = 0 < x_2 < \ldots < x_N = 1$. We call the cell $K_{j-1/2} = [x_{j-1}\,,\,x_j]$ a "box", for $2 \leq j \leq N$. The lenght of the box $K_{j-1/2}$ is $h_{j-1/2} = x_j - x_{j-1}$, with the quasi-uniformity hypothesis $C_m h = h_m \leq h_{j-1/2} \leq h$, where $C_m > 0$ is a constant. We call $h_j = \frac{1}{2}\,(h_{j-1/2} + h_{j+1/2})$ and $h_1 = h_{3/2}/2$, $h_N = h_{N-1/2}/2$. The barycenter of the box $K_{j-1/2}$ is $x_{j-1/2}$. Each grid function $(u_j)_{1 \leq j \leq N}$ is identified with the finite element function $u_h(x)$ belonging to the piecewise affine conforming space $P_c^1(I)$

$$u_h\,(x) = \sum_{j=1}^{N} u_j\,\varphi_j\,(x) \tag{3}$$

where $\varphi_j\,(x)$ is the standard "hat" function centered at node x_j. We denote also by $P_{c,0}^1(I)$ the subspace of $P_c^1(I)$ of the functions $u_h(x)$ such that $u_1 = u_N = 0$. In the sequel, we simply call $P_c^1 = P_c^1(I)$, $P_{c,0}^1 = P_{c,0}^1(I)$. The orthogonal projector onto the functions constant in each box is Π^0 which is given by

$$(\Pi^0 f)_{j-1/2} = \frac{1}{h_{j-1/2}} \int_{x_{j-1}}^{x_j} f(x)\,dx, \qquad f \in L^2\,(I) \tag{4}$$

The standard L^2 and H^1 norms are denoted by $|\ .\ |_{0,I}$, $\|\ .\ \|_{1,I}$, the scalar product in L^2 by $(\ ;\ )_{0,I}$ or simply $(\ ;\ )$. The H^1 semi-norm is $|\ \ |_{1,I}$. In addition, the L^2 mesh-dependent norm is given by

$$|u|_{0,h}^2 = \sum_{j=1}^{N} h_j\,u_j^2 \tag{5}$$

We skip the proof of the following Lemma, which establishes the link between the norms $|u|_{0,I}$, $|u|_{0,h}$, $|\Pi^0 u|$.

Lemma 2.1 *(i) For $u \in P_c^1$, we have $3^{-\frac{1}{2}}|u|_{0,h} \leq |u|_{0,I} \leq |u|_{0,h}$.*
(ii) There exists a constant $C > 0$ independent of h, such that for any $u \in P_{c,0}^1$,

$$C\,h\,|u|_{0,I} \leq |\Pi^0\,u|_{0,I} \leq |u|_{0,I} \tag{6}$$

3 Two box-schemes for the one-dimensional stationary diffusion equation

3.1 Keller's box-scheme

Although rather simple, the interpretation and numerical analysis with finite elements of the classical Keller's box-scheme [22], seems not have been addressed in previous works. We summarize here for convenience some of its properties in the case of the stationary one-dimensional Poisson problem:

$$\left\{\begin{array}{lll} -u_{xx}(x) = f(x) & , & x \in I \\ u(0) = 0 & , & u(1) = 0 \end{array}\right. \tag{7}$$

The design of the box-scheme for (7) is in two steps, [11, 12]. Firstly, (7) is recasted in mixed form

$$\left\{\begin{array}{ll} p_x + f = 0 & (a) \\ p - u_x = 0 & (b) \\ u(0) = u(1) = 0 & (c) \end{array}\right. \tag{8}$$

Secondly, we take the average of (8_a) and (8_b) onto the boxes $K_{j-1/2}$, which gives for the exact solution $(u, p = u_x)$

$$\left\{\begin{array}{l} p(x_j) - p(x_{j-1}) = -h_{j-1/2}\,(\Pi^0 f)_{|j-1/2} \\ h_{j-1/2}\,(\Pi^0 p)_{|j-1/2} - [u(x_j) - u(x_{j-1})] = 0 \\ u(x_1) = u(x_N) = 0 \end{array}\right. \tag{9}$$

Considering now $(u_j\,,\,p_j)_{1\le j\le N}$ approximations of $u(x_j)\,,\,p(x_j)$, (9) defines a numerical scheme if we precise a formula expressing an approximation of $(\Pi^0 p)_{|j-1/2}$ in function of the discrete values p_j. The box-scheme of Keller is characterized by the trapezoidal approximation of the projection $(\Pi^0 p)_{j-1/2}$

$$(\Pi^0 p)_{|j-1/2} = \frac{1}{2}\left(p(x_j) + p(x_{j-1})\right) - \frac{1}{12} h_{j-1/2}^2 p_{xx}(\xi_{j-1/2}), \qquad \xi_{j-1/2} \in K_{j-1/2} \tag{10}$$

Replacing $(\Pi^0 p)_{|j-1/2}$ by $\frac{1}{2}(p_j + p_{j-1})$, suggests to define the box-scheme as

$$\left\{\begin{array}{ll} p_j - p_{j-1} = -h_{j-1/2}\,(\Pi^0 f)_{|j-1/2} & N-1 \text{ equations} \\ \frac{1}{2} h_{j-1/2}\,[p_j + p_{j-1}] - [u_j - u_{j-1}] = 0 & N-1 \text{ equations} \\ u_1 = u_N = 0 & \end{array}\right. \tag{11}$$

(11) is effectively a square system in $(u_j\,,\,p_j)_{1\le j\le N}$ since the number of equations and the number of unknowns are both equal to $2N$. Note also that (11) coincides exactly with (9) when replacing p by its P_c^1-interpolant. The implementation of (11) is carried out by eliminating p_j by static condensation, [11]. Solving the 2×2 system (11) in $(p_{j-1}\,,\,p_j)$ yields for $2 \le j \le N$

$$\left\{\begin{array}{l} p_{j-1} = \dfrac{1}{h_{j-1/2}}(u_j - u_{j-1}) + \dfrac{1}{2} h_{j-1/2}\,(\Pi^0 f)_{j-1/2} \\ p_j = \frac{1}{h_{j-1/2}}(u_j - u_{j-1}) - \frac{1}{2} h_{j-1/2}\,(\Pi^0 f)_{j-1/2} \end{array}\right. \tag{12}$$

By identifying the two values of p_j deduced from $(12)_{j-1/2}$ and $(12)_{j+1/2}$, we obtain the scheme in the unknowns u_j only

$$\left\{\begin{array}{l} -\left[\dfrac{1}{h_{j+1/2}}(u_{j+1}-u_j)-\dfrac{1}{h_{j-1/2}}(u_j-u_{j-1})\right]=\dfrac{1}{2}\Big[h_{j+1/2}(\Pi^0 f)_{j+1/2}+h_{j-1/2}(\Pi^0 f)_{j-1/2}\Big] \\ u_1=u_N=0 \end{array}\right. \tag{13}$$

If $h_{j+1/2}=h_{j-1/2}=h$ we recognize the finite difference compact scheme

$$-\frac{1}{h^2}\left[u_{j+1}+u_{j-1}-2u_j\right]=\frac{1}{2}\left[(\Pi^0 f)_{j+1/2}+(\Pi^0 f)_{j-1/2}\right] \tag{14}$$

This scheme is second order accurate with consistency error given by(we call $u(x)$ a formal solution of $-u_{xx}=f$)

$$\begin{aligned} E_h(u) &= -\frac{1}{h^2}[u(x_{j+1})+u(x_{j-1})-2u(x_j)]-\frac{1}{2}[(\Pi^0 f)_{j+1/2}+(\Pi^0 f)_{j-1/2}] && (15)\\ &= -\frac{1}{12}h^2 u_{xxxx}+O(h^4) && (16)\end{aligned}$$

In [22], Keller proves two error estimates thanks to a finite difference analysis. Translated into the finite element framework stated in Sect.2, these estimates read

$$(a)\qquad |\Pi^0\,\tilde{u}-\Pi^0\,u_h|_{0,I}\le C\,h^2\quad ;\quad (b)\qquad |\Pi^0\,\tilde{p}-\Pi^0\,p_h|_{0,I}\le C\,h^2 \tag{17}$$

where $\tilde{u}\in P^1_{c,0}$, $\tilde{p}\in P^1_c$ denote the standard P^1_c-interpolants of u, p given by

$$\tilde{u}\,(x)=\sum_{j=2}^{N-1}u\,(x_j)\,\varphi_j\,(x)\quad ;\quad \tilde{p}\,(x)=\sum_{j=1}^{N}p\,(x_j)\,\varphi_j\,(x) \tag{18}$$

and where $u_h\in P^1_{c,0}$, $p_h\in P^1_c$ are

$$u_h\,(x)=\sum_{j=2}^{N-1}u_j\,\varphi_j\,(x)\quad ;\quad p_h\,(x)=\sum_{j=1}^{N}p_j\,\varphi_j\,(x) \tag{19}$$

As mentionned by Keller, (17_b) is not an error estimate, because $p_h\in P^1_c\mapsto|\Pi^0\,p_h|_{0,I}$ is not a norm over P^1_c, due to the presence of the oscillating mode $p_j=(-1)^j$, $1\le j\le N$, such that $|\Pi^0 p_h|_{0,I}=0$. In addition, (17_a) is simply a first order error estimate in the L^2 norm $|\cdot|_{0,I}$ due to Lemma 2.1 (ii).
In fact, the interpretation of the box-scheme as a finite-element method allows to perform a more precise numerical analysis. We note that (11) is equivalent to : find $(u_h\,,\,p_h)\in P^1_{c,0}\times P^1_c$ solution of

$$\left\{\begin{array}{lll} (p_{h,x}+f\ ;\ \tilde{v}_h)=0 & \forall\,\tilde{v}_h\in P^0 & (a)\\ (p_h-u_{h,x}\ ;\ \tilde{q}_h)=0 & \forall\,\tilde{q}_h\in P^0 & (b)\end{array}\right. \tag{20}$$

where P^0 stands for the space of piecewise constant functions on the mesh. Taking $\tilde{q}_h=v_{h,x}$, $v_h\in P^1_{c,0}$ in (20_b) and noting that

$$(u_{h,x}\ ;\ v_{h,x})=(p_h\ ;\ v_{h,x})=-(p_{h,x}\ ;\ \Pi^0\,v_h)=(\Pi^0\,f\ ;\ v_h) \tag{21}$$

The box-scheme is therefore equivalent to two decoupled schemes. Firstly, it reduces for u_h to the modified standard finite element method: find $u_h \in P^1_{c,0}$ solution of

$$(u_{h,x}\ ;\ v_{h,x}) = (\Pi^0 f\ ;\ v_h) \qquad \forall\, v_h \in P^1_{c,0} \tag{22}$$

Secondly, p_h has the local expression given by (11), we can rewrite in the form

$$p_{h|K_{j-1/2}}(x) = u_{h,x|K_{j-1/2}} - (\Pi^0 f)_{|j-1/2}\,(x - x_{j-1/2}) \tag{23}$$

We derive now by a standard finite element analysis the well-posedness of scheme (11) as well as *a priori* error estimates. The box-scheme can be rewritten

$$p_{h,x} = -\Pi^0 f\ ;\ u_{h,x} = \Pi^0 p_h \tag{24}$$

and the interpolate $\tilde u$, $\tilde p$ of the exact solution $u(x)$, $p(x) = u_x(x)$ verify

$$\tilde p_x = -\Pi^0 f\ ;\ \tilde u_x = \Pi^0 p \tag{25}$$

Proposition 3.1

(i) The box-scheme (11) has a unique solution $(u_h\,,\,p_h) \in P^1_{c,0} \times P^1_c$ verifying

$$\|u_h\|_{1,I} + \|p_h\|_{1,I} \le C\,|f|_{0,I} \tag{26}$$

(ii) If (u,p) is the solution of (8), $(\tilde u, \tilde p)$, their P^1_c-interpolants, then the following a priori error estimates hold, (C is a generic constant independent of h)

$$\left\{\begin{array}{l} (a)\quad \|u-u_h\|_{1,I} \le C\,h\,|f|_{0,I} \qquad (b)\quad |u-u_h|_{0,I} \le C\,h^2\,\|f\|_{1,I} \\ (c)\quad \|p-p_h\|_{1,I} \le C\,h\,\|f\|_{1,I} \quad ; \qquad (d)\quad |\Pi^0 p - \Pi^0 p_h|_{0,I} \le C\,h^{3/2}\,|f|_{1,I} \\ \qquad (e)\quad |\Pi^0 \tilde p - \Pi^0 p_h|_{0,I} \le C\,h^2\,|u_{xxx}|_{0,I} \end{array}\right. \tag{27}$$

Proof: (i) results simply from (22) and (23). A standard error analysis in $P^1_{c,0}$ yields estimate (27_a). Estimate (27_b) results of the Aubin-Nitsche argument, together with the fact that $|f - \Pi^0 f|_{0,I} \le C\,h\,|f|_{1,I}$. This estimate implies obviously the first Keller's estimate in (17_a). Estimate (27_c) results directly from (23). We prove now the two last estimates. Note that the last estimate is precisely the one given by Keller in (17_b), using finite-differencing arguments. For the estimate (27_d), we have the identities

$$\begin{aligned} |\Pi^0 p - \Pi^0 p_h|^2_{0,I} &= (p - p_h\ ;\ \Pi^0 p - \Pi^0 p_h)_{0,I} = (\tilde u_x - u_{h,x}\ ;\ p - p_h) = -(\tilde u - u_h; p_x - p_{h,x}) \\ &= (\tilde u - u_h; f - \Pi^0 f) \le (|\tilde u - u|_{0,I} + |u - u_h|_{0,I})|f - \Pi^0 f|_{0,I} \le C h^3 \|f\|^2_{1,I} \end{aligned}$$

which gives the result. For (27_e), we decompose $|\,\Pi^0 \tilde p - \Pi^0 p_h\,|^2_{0,I}$ into a numerical quadrature error part and a consistency part

$$|\,\Pi^0 \tilde p - \Pi^0 p_h\,|^2_{0,I} = \underbrace{(\Pi^0 \tilde p - \Pi^0 p\ ;\ \Pi^0 \tilde p - \Pi^0 p_h)}_{(I)} + \underbrace{(\Pi^0 p - \Pi^0 p_h\ ;\ \Pi^0 \tilde p - \Pi^0 p_h)}_{(II)} \tag{28}$$

The trapezoidal error estimate (10) yields

$$\left|\Pi^0 \tilde p - \Pi^0 p\right|_{0,I} \le C\,h^2\,|p|_{2,I} \tag{29}$$

which gives

$$|(I)| \leq C\,h^2\,|p_{xx}|_{0,I}|\Pi^0\,\tilde{p} - \Pi^0\,p_h|_{0,I} \tag{30}$$

The term (II) vanishes because

$$\begin{aligned} (II) &= (u_x - u_{h,x}\ ;\ \Pi^0\,\tilde{p} - \Pi^0\,p_h)_{0,I} = (\tilde{u}_x - \tilde{u}_{h,x}\ ;\ \Pi^0\,\tilde{p} - \Pi^0\,p_h)_{0,I} \\ &= (\tilde{u}_x - u_{h,x}\ ;\ \tilde{p} - p_h) = -(\tilde{u} - u_h\ ;\ \tilde{p}_x - p_{h,x}) = 0 \end{aligned}$$

Finally, the error estimate (27_e) results only from the second order quadrature (30). ■

3.2 A fourth order box-scheme

Starting again from (9), we build an higher order box-scheme by replacing the trapezoidal quadrature formula (10) by the fourth order formula deduced from the Euler-MacLaurin series. Recall that this series reads, for any regular function F defined on $[a, b]$

$$\frac{1}{b-a}\int_a^b F(t)dt = \frac{1}{2}(F(a) + F(b)) - \sum_{i=1}^{m} \frac{B_{2i}}{(2i)!}(b-a)^{2i-1}(F^{(2i-1)}(b) - F^{(2i-1)}(a)) - E_m \tag{31}$$

with an error term given by

$$E_m = \frac{B_{2m+2}(b-a)^{2m+2}}{(2m+2)!}F^{(2m+2)}(\xi), \qquad a < \xi < b \tag{32}$$

The B_i are the Bernoulli numbers defined by the serie

$$\frac{t}{e^t - 1} = \sum_{i=0}^{+\infty} \frac{B_i}{i!}t^i \tag{33}$$

Using the fourth order quadrature formula on the box $K_{j-1/2} = [x_{j-1}, x_j]$ deduced from (31) yields

$$(\Pi^0 p)_{j-1/2} = \frac{1}{2}(p(x_j) + p(x_{j-1}) - \frac{1}{12}h_{j-1/2}(p_x(x_j) - p_x(x_{j-1})) - E^p_{j-1/2} \tag{34}$$

with an error

$$E^p_{j-1/2} = \frac{B_4 h^4_{j-1/2}}{4!}p^{(4)}(\xi_{j-1/2}), \qquad x_{j-1} < \xi_{j-1/2} < x_j \tag{35}$$

Since $p_x(x) = -f(x)$, we deduce

$$(\Pi^0 p)_{j-1/2} = \frac{1}{2}(p(x_j) + p(x_{j-1}) + \frac{1}{12}h_{j-1/2}(f(x_j) - f(x_{j-1})) - E^p_{j-1/2} \tag{36}$$

Replacing now in (9) $(\Pi^0 p)_{j-1/2}$ by the approximated value

$$\frac{1}{2}(p_j + p_{j-1}) + \frac{1}{12}h_{j-1/2}(f(x_j) - f(x_{j-1})) = \frac{1}{2}(p_j + p_{j-1}) + \frac{1}{12}h^2_{j-1/2}(\Pi^0 f')_{j-1/2} \tag{37}$$

allows to write the following box-scheme

$$\left\{ \begin{array}{l} p_j - p_{j-1} = -h_{j-1/2}\,(\Pi^0\,f)_{|j-1/2} \\ \frac{1}{2}\,h_{j-1/2}\,[p_j + p_{j-1}] - [u_j - u_{j-1}] = -\frac{1}{12}h^3_{j-1/2}(\Pi^0 f')_{j-1/2} \\ u_1 = u_N = 0 \end{array} \right. \tag{38}$$

Solving (38) in $(p_{j-1}\,,\,p_j)$ gives

$$\left\{\begin{array}{l} p_{j-1}=(u_j-u_{j-1})/h_{j-1/2}+\frac{1}{2}\,h_{j-1/2}\,(\Pi^0 f)_{j-1/2}-\frac{1}{12}h^2_{j-1/2}(\Pi^0 f')_{j-1/2} \\ p_j=(u_j-u_{j-1})/h_{j-1/2}-\frac{1}{2}\,h_{j-1/2}\,(\Pi^0 f)_{j-1/2}-\frac{1}{12}h^2_{j-1/2}(\Pi^0 f')_{j-1/2} \end{array}\right. \tag{39}$$

The scheme in u_j after elimination of p_j is now

$$\left\{\begin{array}{l} -\left[\dfrac{1}{h_{j+1/2}}\,(u_{j+1}-u_j)-\dfrac{1}{h_{j-1/2}}\,(u_j-u_{j-1})\right]=\dfrac{1}{2}\,h_{j+1/2}\,(\Pi^0 f)_{j+1/2}+\dfrac{1}{2}h_{j-1/2}\,(\Pi^0 f)_{j-1/2} \\ -\frac{1}{12}h^2_{j+1/2}(\Pi^0 f')_{j+1/2}+\frac{1}{12}h^2_{j-1/2}(\Pi^0 f')_{j-1/2} \\ u_1=u_N=0 \end{array}\right. \tag{40}$$

On an equally spaced mesh, the scheme reduces to the following finite difference compact scheme

$$-\frac{1}{h^2}\,[u_{j+1}+u_{j-1}-2u_j]=\frac{1}{2}\,[(\Pi^0 f)_{j+1/2}+(\Pi^0 f)_{j-1/2}]-\frac{1}{12}h[(\Pi^0 f')_{j+1/2}-(\Pi^0 f')_{j-1/2}] \tag{41}$$

which is fourth order accurate with a consistency error

$$E_h(u)=\frac{1}{720}h^4 u^{(6)}+O(h^6) \tag{42}$$

Note that replacing $(\Pi^0 f)_{j+1/2}$, (resp. $(\Pi^0 f')_{j+1/2}$) by a fourth order (resp. third order) formula, does not modify the fourth order of the scheme. Let us state now the formulation of the box-scheme (38) as a finite element scheme for $u_h(x)=\sum_j u_j\varphi_j(x)$ combined with a reconstruction formula for $p_h(x)=\sum_j p_j\varphi_j(x)$.

Proposition 3.2 *(a) The box-scheme (38) is equivalent to the twofold finite element scheme:*
(i) find $u_h(x)\in P^1_{c,0}$ such that

$$(u_{h,x};v_{h,x})=(\Pi^0 f;v_h)+\frac{1}{12}(\delta_h(\Pi^0 f');v_{h,x})_{0,I} \tag{43}$$

where $\delta_h(x)=\sum_{j=2}^N h^2_{j-1/2}\mathbb{1}_{K_{j-1/2}}(x)$.
(ii) reconstruct $p_h\in P^1_c$ by

$$p_{h|K_{j-1/2}}(x)=(\Pi^0\,p_h)_{|j-1/2}+p_{h,x|K_{j-1/2}}(x-x_{j-1/2}) \tag{44}$$

with $(\Pi^0\,p_h)_{|j-1/2}=u_{h,x|K_{j-1/2}}-\frac{1}{12}h^2_{j-1/2}(\Pi^0 f')_{j-1/2}$ and $p_{h,x|K_{j-1/2}}=-(\Pi^0 f')_{j-1/2}$.
(b) The following superconvergence estimates between the P^1_c interpolants $\tilde{u}$, $\tilde{p}$ of u,p and u_h, p_h, hold

$$(i)\qquad |\tilde{u}-u_h|_{0,I}\le Ch^4|u|_{5,I} \tag{45}$$

$$(ii)\qquad |\Pi^0\tilde{p}-\Pi^0 p_h|_{0,I}\le Ch^4|u|_{5,I} \tag{46}$$

Note that, due to Lemma 2.1, *(ii)* is a third order estimate in $|\tilde{p}-p_h|_{0,I}$.
Proof: The box-scheme (38) is simply: find $(u_h,p_h)\in P^1_{c,0}\times P^1_c$ solution of

$$\left\{\begin{array}{lll} (p_{h,x}+f\ ;\ \tilde{v}_h)=0 & \forall\,\tilde{v}_h\in P^0 & (a) \\ (p_h-u_{h,x}+\frac{1}{12}\delta_h f'\ ;\ \tilde{q}_h)=0 & \forall\,\tilde{q}_h\in P^0 & (b) \end{array}\right. \tag{47}$$

It could be noticed in (47) that the constitutive law $p = u_x$ is no longer verified in the mean at the discrete level, contrary to (20). This scheme is equivalent to the identities:

$$p_{h,x} = -\Pi^0 f \qquad ; \qquad u_{h,x} = \Pi^0 p_h + \frac{1}{12}\delta_h \Pi^0 f' \tag{48}$$

Selecting $\tilde{q}_h = v_{h,x}$ in (47_b) gives (43). The reconstruction formula (44) just follows from (47). Let us check now (46). As in (28), we decompose

$$|\,\Pi^0\,\tilde{p} - \Pi^0\,p_h\,|_{0,I}^2 = \underbrace{(\Pi^0\,\tilde{p} - \Pi^0\,p\ ;\ \Pi^0\,\tilde{p} - \Pi^0\,p_h)}_{(I)} + \underbrace{(\Pi^0\,p - \Pi^0\,p_h\ ;\ \Pi^0\,\tilde{p} - \Pi^0\,p_h)}_{(II)} \tag{49}$$

We have

$$\begin{aligned} (II) &= (\Pi^0\,p - u_{h,x} + \frac{1}{12}\delta_h \Pi^0 f'\ ;\ \Pi^0\,\tilde{p} - \Pi^0\,p_h) \\ &= (\tilde{u}_x - u_{h,x} + \frac{1}{12}\delta_h \Pi^0 f'\ ;\ \Pi^0\,\tilde{p} - \Pi^0\,p_h) \end{aligned}$$

Using the relation

$$\frac{1}{12}\delta_h(\Pi^0 f') = \Pi^0\,p - \Pi^0\,\tilde{p} + E^p \qquad , \qquad E^p = \sum_{j=2}^{N} E^p_{j-1/2} \mathbb{1}_{K_{j-1/2}}(x) \tag{50}$$

we obtain finally $(I) + (II) = (E^p(x), \Pi^0\,\tilde{p} - \Pi^0\,p_h)$, from which we deduce easily (46). For (45), we start from the relations $\tilde{u}_x = \Pi^0 p$, $u_{h,x} = \Pi^0 p_h + \frac{1}{12}\delta_h(\Pi^0 f') = \Pi^0 p - (\Pi^0 \tilde{p} - \Pi^0 p_h) + E^p$. Therefore $\tilde{u}_x - u_{h,x} = \Pi^0 \tilde{p} - \Pi^0 p_h - E^p$, giving $|\tilde{u}_x - u_{h,x}|_{0,I} \leq Ch^4 |u|_{5,I}$, and finally (45) by the Poincaré inequality. ■

4 A first order box-scheme for the stationary convection-diffusion equation

4.1 Presentation of the scheme

We recall in this section the extension of Keller's scheme to the standard stationary convection-diffusion equation with source term f proposed by Courbet in [9]. Although this scheme is only first order accurate in the finite difference sense, its interest lies in its design, which can be extended to more complex situations. Moreover, we prove that it is equivalent to a non-standard SUPG scheme justiciable of an error estimate in a suitable energy norm of order $h^{3/2}$, when $\varepsilon = O(h)$. Let us consider the convection-diffusion problem

$$\left\{ \begin{array}{l} cu_x - \varepsilon u_{xx} = f \qquad x \in]0,1[\quad , \quad c \in \mathbb{R} \quad , \quad \varepsilon > 0 \\ u\,(0) = u\,(1) = 0 \end{array} \right. \tag{51}$$

We rewrite (51) in mixed form as

$$\left\{ \begin{array}{ll} cu_x + p_x = f & (I_a) \\ p + \varepsilon u_x = 0 & (I_b) \\ u\,(0) = u\,(1) = 0 & (I_c) \end{array} \right. \tag{52}$$

$f(u)=cu$ is the convective flux and $p=-\varepsilon u_x$ the diffusive flux. As in Sect.3, the exact solution $(u, p=-\varepsilon u_x)$ verifies the discrete equations obtained by averaging $(52)_a$,$(52)_b$ onto the boxes $K_{j-1/2}$

$$\left\{\begin{array}{l} c\,(u(x_j)-u\,(x_{j-1}))+p\,(x_j)-p\,(x_{j-1})=h_{j-1/2}\;(\Pi^0\,f)_{|j-1/2} \\ h_{j-1/2}\,(\Pi^0 p)_{|j-1/2}+\varepsilon\,(u\,(x_j)-u\,(x_{j-1}))=0 \\ u\,(x_1)=u\,(x_N)=0 \end{array}\right. \tag{53}$$

The box-scheme is now defined from (53), by approximating $(\Pi^0\,p)_{|j-1/2}$ in function of p_j. The class of schemes we consider *a priori* is given by the upwind approximation of $\Pi^0 p$ in the form

$$(\Pi^0 p)_{j-1/2}=\frac{1}{2}\,[p_{j-1}+p_j]-D_{p,j-1/2}\,[p_j-p_{j-1}] \tag{54}$$

$D_{p,j-1/2}$ is an upwinding coefficient in the boxes $K_{j-1/2}$ whose meaning will appear later. As for the standard box-scheme (11), they are in (53) $2N$ unknowns u_j , p_j and $2N$ equations. The resulting scheme writes

$$\left\{\begin{array}{l} c[u_j-u_{j-1}]+[p_j-p_{j-1}]=h_{j-1/2}(\Pi^0 f)_{j-1/2} \\ \frac{1}{2}\,(p_j+p_{j-1})-D_{p,j-1/2}\,(p_j-p_{j-1})+\frac{\varepsilon}{h_{j-1/2}}\,(u_j-u_{j-1})=0 \\ u_1=u_N=0 \end{array}\right. \tag{55}$$

With the notation $\Delta\,u_{j-1/2}=(u_j-u_{j-1})/h_{j-1/2}$ the resolution of (55) with respect to p_{j-1} , p_j yields

$$\left\{\begin{array}{l} p_{j-1}=-\left[\varepsilon-c\left(\frac{1}{2}-D_{p,j-1/2}\right)\,h_{j-1/2}\right]\,\Delta\,u_{j-1/2}+[-\frac{1}{2}+D_{p,j-1/2}]\,h_{j-1/2}\,(\Pi^0 f)_{j-1/2} \\ p_j=-\left[\varepsilon+c\left(\frac{1}{2}+D_{p,j-1/2}\right)\,h_{j-1/2}\right]\,\Delta\,u_{j-1/2}+[\frac{1}{2}+D_{p,j-1/2}]\,h_{j-1/2}\,(\Pi^0 f)_{j-1/2} \end{array}\right. \tag{56}$$

Identifying the two values of p_j given by (56) in $K_{j-1/2}$, $K_{j+1/2}$ gives the scheme in (u_j) only

$$\begin{aligned} \frac{c}{2h_j}\,(u_{j+1} &- u_{j-1})-\frac{1}{h_j}\left[(\varepsilon+c\,h_{j+1/2}D_{p,j+1/2})\,\Delta\,u_{j+1/2}-(\varepsilon+c\,h_{j-1/2}D_{p,j-1/2})\,\Delta\,u_{j-1/2}\right] \\ &= \frac{1}{h_j}\left[\left(\frac{1}{2}-D_{p,j+1/2}\right)\,h_{j+1/2}\,(\Pi^0 f)_{j+1/2}+\left(\frac{1}{2}+D_{p,j-1/2}\right)\,h_{j-1/2}\,(\Pi^0 f)_{j-1/2}\right] \end{aligned}$$

The scheme can be rewritten in function of $\Delta\,u_{j-1/2}$, $\Delta\,u_{j+1/2}$ only as

$$\begin{aligned} &- \left\{[\varepsilon+(D_{p,j+1/2}-1/2)\,c\,h_{j+1/2}]\Delta\,u_{j+1/2}-[\varepsilon+(D_{p,j-1/2}+1/2)c\,h_{j-1/2}]\Delta\,u_{j-1/2}\right\} \\ &= \left(\frac{1}{2}-D_{p,j+1/2}\right)\,h_{j+1/2}\,(\Pi^0 f)_{j+1/2}+\left(\frac{1}{2}+D_{p,j-1/2}\right)\,h_{j-1/2}\,(\Pi^0 f)_{j-1/2} \end{aligned} \tag{57}$$

In addition, multiplying $(55)_2$ by c and replacing $c\,(u_j-u_{j-1})$ by its value deduced from $(55)_1$ gives the following scheme in (p_j), consistent with the equation $c\,p-\varepsilon\,p_x=-\varepsilon\,f$

$$\frac{1}{2}\,c\,(p_j+p_{j-1})-\left(ch_{j-1/2}\,D_{p,j-1/2}+\varepsilon\right)\,\frac{(p_j-p_{j-1})}{h_{j-1/2}}=-\varepsilon\,(\Pi^0 f)_{j-1/2} \tag{58}$$

The quantity $ch_{j-1/2}D_{p,j-1/2}+\varepsilon$ appears as the total diffusion of the scheme and $ch_{j-1/2}D_{p,j-1/2}$ as an artificial diffusion coefficient. We suppose from now on that $cD_{p,j-1/2}\geq 0$ and we give a sufficient condition on the coefficients $D_{p,j-1/2}$ in order for the box-scheme to verify a maximum principle for u_j, p_j. More precisely, we consider scheme (55) with $f=0$ and boundary conditions $u_1 = u(0) = \alpha$, $u_N = u(1) = \beta$. The exact solution $u(x)$, $p(x) = u_x(x)$ exhibits an exponential boundary layer given by (2). A traditional requirement for schemes approximating this problem is to verify the maximum principle. Here, it consists simply in the monotonicity of the two sequences u_j, p_j. Let us define the cell-Peclet number $Pe_{j-1/2} \in [0, +\infty[$ for $c \in \mathbb{R}$, $\varepsilon > 0$ by

$$Pe_{j-1/2} = \frac{|c| h_{j-1/2}}{2\varepsilon} \tag{59}$$

Proposition 4.1 *(i) Suppose u_j, p_j are solution of the box-scheme (55) with $f=0$ and nonhomogeneous Dirichlet boundary conditions $u_1 = \alpha$, $u_N = \beta$. A necessary and sufficient condition for the p_j to have same sign is that for any $j = 2,..,N$*

$$\varepsilon + cD_{p,j-1/2} h_{j-1/2} \geq \frac{|c|}{2} h_{j-1/2} \tag{60}$$

In that case, u_j, p_j are monotone sequences. The value of $D_{p,j-1/2}$ for which $|D_{p,j-1/2}|$ is minimum in (60) is

$$D_{p,j-1/2} = \frac{1}{2}\,\mathrm{sgn}(c)\max\left(0\ ;\ 1 - \frac{1}{Pe_{j-1/2}}\right) \tag{61}$$

where $\mathrm{sgn}(c)$ *is the sign function.*
(ii) If $h_j = h$ and $D_{p,j-1/2} = D_p$, the consistency error in the finite difference sense of scheme (57) with respect to the equation $cu_x - \varepsilon u_{xx} = f$ is

$$E_h(x) = -h\varepsilon\, D_p u_{xxx} + O\left(h^2\right) \tag{62}$$

(iii) The consistency error of scheme (58) with respect to $cp - \varepsilon\, p_x = -\varepsilon f$, is

$$\bar{E}_h\left(x\right) = -hcD_p p_x + O(h^2) \tag{63}$$

Proof: *(i)* Suppose $f=0$. Due to $(55)_1$, the sequence u_j is monotone if and only if so is p_j. Taking $f=0$ in (58), and under the hypothesis $cD_{p,j-1/2} \geq 0$, all the p_j have same sign if and only if (60) is true. Equation (57) ensures now that the coefficients before $\Delta u_{j+1/2}$, $\Delta u_{j-1/2}$ have the same sign. Therefore (u_j) is monotone under condition (60), and so is (p_j). The minimum value of $|D_{p,j-1/2}|$ is clearly obtained with (61).
(ii) We suppose now $h_{j-1/2} = h$, $D_{p,j-1/2} = D_p$, $2 \leq j \leq N$. This allows to perform a standard finite difference analysis. Scheme (57) rewrites in the form of a three points compact scheme for the convection-diffusion equation

$$\alpha_{-1}\, u_{j-1} + \alpha_0\, u_j + \alpha_1\, u_{j+1} = (\frac{1}{2} - D_p)(\Pi^0 f)_{j+1/2} + (\frac{1}{2} + D_p)(\Pi^0 f)_{j-1/2} \tag{64}$$

The coefficients α_1, α_0, α_{-1} are

$$\alpha_1 = \frac{1}{h}\left[\frac{c}{2} - \left(\frac{\varepsilon}{h} + c\,D_p\right)\right]\ ;\ \alpha_0 = \frac{2}{h}\left(\frac{\varepsilon}{h} + c\,D_p\right)\ ;\ \alpha_{-1} = \frac{1}{h}\left[-\frac{c}{2} - \left(\frac{\varepsilon}{h} + c\,D_p\right)\right] \tag{65}$$

The consistency error is, [27]

$$E_h\, u(x_j) = \left(L_h - R_h \circ L\right) u(x_j) \tag{66}$$

where L_h and R_h are the finite difference operators defined by $L_h\, u_{|j} = \alpha_{-1}\, u_{j-1} + \alpha_0\, u_j + \alpha_1\, u_{j+1}$, $R_h\, f_{|j} = (\frac{1}{2} - D_p)(\Pi^0 f)_{j+1/2} + (\frac{1}{2} + D_p)(\Pi^0 f)_{j-1/2}$. If $p_h(\xi) = \alpha_{-1} e^{-i\theta} + \alpha_0 + \alpha_1 e^{i\theta}$, $r_h(\xi) = (\frac{1}{2} - D_p)(e^{-i\theta} - 1)/(i\theta) + (\frac{1}{2} + D_p)(1 - e^{-i\theta})/(i\theta)$ ($\theta = \xi h$) are the symbols of the operators L_h, R_h, and $p(\xi) = ic\xi + \varepsilon\xi^2$ is the symbol of the convection-diffusion operator $Lu(x) = cu_x - \varepsilon u_{xx}$, then we find that the symbol of the consistency error $E_h(x)$ is $e_h\,(\xi) = p_h\,(\xi) - r_h\,(\xi)\, p\,(\xi)$, with

$$e_h\,(\xi) = i\,\varepsilon\, h\xi^3\, D_p + O\,(h^2) \tag{67}$$

which is (62).
(iii) We proceed in the same way. The operator L is $Lp = cp - \varepsilon p_x$ with symbol $p(\xi) = c - \varepsilon i\xi$. The symbol of the finite difference operator $L_h p_{|j}$ in the r.h.s. of (58) is ($\theta = \xi h$), $p_h(\xi) = c(1 + e^{-i\theta}/2) - (cD_p + \varepsilon/h)(1 - e^{-i\theta})$. The symbol of the interpolation operator acting on $F = -\varepsilon f$ is $r_h(\xi) = \frac{1}{2}(1 - e^{-i\theta})$. The symbol of the consistency error $\bar{E}_h(x) = (L_h - R_h \circ L)p_j$ is therefore

$$\bar{e}_h(\xi) = -ihc\xi D_p + O(h^2) \tag{68}$$

which gives (63). ∎

Remark: It results from Prop 4.1 that if $D_p \neq 0$, the box-scheme is only first order accurate in the finite-difference sense with respect to the two equations $cu_x - \varepsilon u_{xx} = f$ and $cp - \varepsilon p_x = -\varepsilon f$. If $D_p = 0$, it is second order accurate.

4.2 Comparison with the SUPG method

In this section, we establish a link between the box-scheme (53) and the standard SUPG method of T.J.R. Hughes *et al.*,[3, 18], C. Johnson *et al.* [21, 19, 26, 20] for the convection-diffusion equation. In its simpler form, this method reads : find $u_h \in P^1_{c,0}$ such that for any $v_h \in P^1_{c,0}$

$$(cu_{h,x}\ ;\ v_h + \delta\, c\, v_{h,x}) + \varepsilon\,(u_{h,x}\ ;\ v_{h,x}) = (f\ ;\ v_h + \delta c\, v_{h,x}) \tag{69}$$

where $\delta = O(h)$ is an upwind parameter. The standard error estimate is usually derived for the stationary convection-diffusion equation with an absorption term σu, with $\sigma > 0$, that is

$$\left\{ \begin{array}{l} \sigma u + cu_x - \varepsilon u_{xx} = f \\ u(0) = u(1) = 0 \end{array} \right. \tag{70}$$

Recall that the equation without absorption term $c\tilde{u}_x - \varepsilon \tilde{u}_{xx} = g$ is transformed into an equation with an absorption term σu by the change of unknown $u(x) = e^{-\sigma x}\tilde{u}(x)$ with $\sigma c > 0$ and $|\sigma|$ sufficiently small. The SUPG method for (70) consists simply in: find $u_h \in P^1_{c,0}$ such that for any $v_h \in P^1_{c,0}$

$$(\sigma u_h + cu_{h,x}\ ;\ v_h + \delta\, c\, v_{h,x}) + \varepsilon\,(u_{h,x}\ ;\ v_{h,x}) = (f\ ;\ v_h + \delta c\, v_{h,x}) \tag{71}$$

The energy norm associated with (71) is

$$|\!|\!| u |\!|\!|^2 = \sigma |u|^2_{0,I} + \delta |cu_x|^2_{0,I} + \varepsilon |u_x|^2_{0,I} \tag{72}$$

Note that $|\!|\!| u |\!|\!|$ controls the L^2 norm $|u|_{0,I}$ uniformly when $\varepsilon \to 0$, only when $\sigma > 0$. The main interest of the SUPG method is that one get, by choosing $\delta = O(h)$, an error estimate in the

energy norm in the form $\|u - u_h\| \leq C h^{3/2}$ uniformly with respect to $\varepsilon = O(h)$. This represents a gain of $h^{1/2}$ in comparison with the standard Galerkin method. In two dimensions, one can replace the diffusion ε by $\varepsilon_m = \max(\varepsilon, C h^{3/2})$, in order to keep an $O(h^{3/2})$ error estimate, still uniformly when $\varepsilon \to 0$ even in the crosswind direction, [20, 32].
Here, we are going to prove that the box-scheme (55) is nothing but a variant of the SUPG method for the unknown u, coupled with a reconstruction formula of the flux p. Note that this last property is unusual in the classical SUPG finite element method. We define the upwinding function $d_p(x)$[2], which is constant in each box $K_{j-1/2}$, by

$$d_p(x) = \sum_{j=2}^{N} D_{p,j-1/2}\, h_{j-1/2}\, \mathbb{1}_{K\,j-1/2}(x) \tag{73}$$

We obtain the following SUPG form of the box-scheme (55):

Proposition 4.2 *The box-scheme scheme (55) is equivalent to*
(i) the modified SUPG method for u_h : find $u_h \in P^1_{c,0}$ such that for $v_h \in P^1_{c,0}$.

$$(c u_{h,x}\ ;\ v_h + d_p\, v_{h,x}) + \varepsilon\,(u_{h,x}\ ;\ v_{h,x}) = (\Pi^0 f\ ;\ v_h + d_p\, v_{h,x}) \tag{74}$$

(ii) the local reconstruction formula in p_h

$$p_{h|K_{j-1/2}} = (\Pi^0\, p_h)_{|K_{j-1/2}} + p_{h,x|K_{j-1/2}}(x - x_{j-1/2}) \tag{75}$$

with

$$(\Pi^0\, p_h)_{|K_{j-1/2}} = -(\varepsilon + c d_{p,j-1/2})\, u_{h,x|K_{j-1/2}} + d_{p,j-1/2}\, (\Pi^0 f)_{j-1/2} \tag{76}$$

$$p_{h,x|K_{j-1/2}} = (\Pi^0 f)_{j-1/2} - c u_{h,x|K_{j-1/2}} \tag{77}$$

Proof: The box-scheme (55) can be rewritten as the mixed Petrov-Galerkin method in $(u_h, p_h) \in P^1_{c,0} \times P^1_c$

$$\left\{ \begin{array}{ll} (c u_{h,x}\ ;\ \tilde{v}_h) + (p_{h,x}\ ;\ \tilde{v}_h) = (f\ ;\ \tilde{v}_h) & \forall\, \tilde{v}_h \in P^0 \\ (\Pi^0\, p_h - d_p\, p_{h,x}\ ;\ \tilde{q}_h) + \varepsilon\,(u_{h,x}\ ;\ \tilde{q}_h) = 0 & \forall\, \tilde{q}_h \in P^0 \end{array} \right. \tag{78}$$

Taking $v_h \in P^1_{c,0}$, we deduce from (78_2) with $\tilde{q}_h = v_{h,x}$

$$\begin{aligned} 0 &= (\Pi^0\, p_h - d_p\, p_{h,x}\ ;\ v_{h,x}) + \varepsilon\,(u_{h,x}\ ;\ v_{h,x}) \\ &= -(p_{h,x}\ ;\ \Pi^0\, v_h) - (d_p\, p_{h,x}\ ;\ v_{h,x}) + \varepsilon\,(u_{h,x}\ ;\ v_{h,x}) \\ &= (c u_{h,x}\ ;\ \Pi^0\, v_h) - (f\ ;\ \Pi^0\, v_h) - (d_p\, p_{h,x}\ ;\ v_{h,x}) + \varepsilon\,(u_{h,x}\ ;\ v_{h,x}) \\ &= (c u_{h,x}\ ;\ v_h) - (\Pi^0 f\ ;\ v_h) - (d_p\, p_{h,x}\ ;\ v_{h,x}) + \varepsilon\,(u_{h,x}\ ;\ v_{h,x}) \end{aligned} \tag{79}$$

Taking now $\tilde{v}_h = d_p\, v_{h,x} \in P^0$ as test function in (78_1), we get $(d_p\, p_{h,x}\ ;\ v_{h,x}) = (\Pi^0 f\ ;\ d_p\, v_{h,x}) - (c u_{h,x}\ ;\ d_p\, v_{h,x})$. Replacing this value in (79) yields

$$(c u_{h,x}\ ;\ v_h + d_p\, v_{h,x}) + \varepsilon\,(u_{h,x}\ ;\ v_{h,x}) = (\Pi^0 f\ ;\ v_h + d_p\, v_{h,x}) \tag{80}$$

[2] $d_p(x)$ should be denoted $d_{p,h}(x)$, in order to indicate the mesh dependency.

This is the standard P^1 streamline-upwind Petrov-Galerkin method with source term $\Pi^0 f$ and upwinding function $d_p(x)$. The flux p_h is now given by the local affine formula in each cell $p_{h|j-1/2} = (\Pi^0 p_h)_{j-1/2} + p_{h,x|j-1/2}(x - x_{j-1/2})$. Formulas (76, 77) result immediately from (78). ■

Taking advantage of this form, we can follow the same sketch of proof as for the one of (71), to prove an error estimate for the box-scheme (78) for the convection-diffusion equation with an absorption term σu. More precisely, we restrict ourselves for simplicity to the adimensional form of the model problem (70) when $c > 0$, [21, 32]

$$\left\{\begin{array}{ll} u + u_x - \varepsilon u_{xx} = f & x \in]0,1[\quad , \quad \varepsilon > 0 \\ u(0) = u(1) = 0 & \end{array}\right. \tag{81}$$

The box-scheme is a discretization of the mixed form

$$\left\{\begin{array}{l} u + u_x + p_x = f \\ p + \varepsilon u_x = 0 \\ u(0) = u(1) = 0 \end{array}\right. \tag{82}$$

It reads

$$\left\{\begin{array}{l} h_{j-1/2}\dfrac{u_j + u_{j-1}}{2} + [u_j - u_{j-1}] + [p_j - p_{j-1}] = h_{j-1/2}(\Pi^0 f)_{j-1/2} \\ \frac{1}{2}(p_j + p_{j-1}) - D_{p,j-1/2}(p_j - p_{j-1}) = -\frac{\varepsilon}{h_{j-1/2}}(u_j - u_{j-1}) \\ u_1 = u_N = 0 \end{array}\right. \tag{83}$$

The cell-Peclet number is $Pe_{j-1/2} = h_{j-1/2}/2\varepsilon \geq C_m h/2\varepsilon$. The upwinding function is given by

$$d_p(x) = \sum_{j=2}^{N} D_{p,j-1/2}\, h_{j-1/2}\, \mathbb{1}_{K\, j-1/2}(x), \quad D_{p,j-1/2} = \frac{1}{2}\max(0, 1 - \frac{1}{Pe_{j-1/2}}) \tag{84}$$

As in Prop. 4.2, (83) is equivalent to:
(i) the SUPG scheme for $u_h \in P^1_{c,0}$

$$(\Pi^0 u_h + u_{h,x}; v_h + d_p v_{h,x}) + \varepsilon\,(u_{h,x}\ ;\ v_{h,x}) = (\Pi^0 f\ ;\ v_h + d_p v_{h,x}) \qquad \forall v_h \in P^1_{c,0} \tag{85}$$

(ii) the local reconstruction formula for $p_h \in P^1_c$ in function of u_h

$$p_{h|j-1/2} = (\Pi^0 p_h)_{|j-1/2} + p_{h,x|j-1/2}(x - x_{j-1/2}) \tag{86}$$

with

$$(\Pi^0 p_h)_{|j-1/2} = -(\varepsilon + d_{p,j-1/2})\, u_{h,x|j-1/2} + d_{p,j-1/2}\, \Pi^0 (f - u_h)_{|j-1/2} \tag{87}$$

and

$$p_{h,x|j-1/2} = \Pi^0 (f - u_h)_{|j-1/2} - u_{h,x|j-1/2} \tag{88}$$

The numerical analysis follows now the lines of classical works on SUPG methods, [3, 18, 19, 20, 21, 26]. Defining for $(u, v) \in (H^1_0)^2$ the mesh-dependent bilinear form B_h and linear from L_h by

$$B_h(u, v) = (\Pi^0 u + u_x\ ;\ v + d_p v_x) + \varepsilon\,(u_x\ ;\ v_x) \quad , \quad L_h(v) = (\Pi^0 f; v + d_p v_x) \tag{89}$$

the scheme rewrites
(i) find $u_h \in P^1_{c,0}$ solution of

$$B_h(u_h, v_h) = L_h(v_h) \qquad \forall v_h \in P^1_{c,0} \tag{90}$$

(ii) reconstruct $p_h \in P^1_c$ by (86-88).
Defining the associated mesh-dependent norm $\|\!|\ .\ \|\!|_h$ by

$$\|\!|u\|\!|_h^2 = |\Pi^0 u|^2_{0,I} + ((\varepsilon + d_p)u_x, u_x)_{0,I} \tag{91}$$

the stability estimate for the form B reads

Lemma 4.1 *For $u \in H^1_0$, there exists $0 < C_0 < 1$ such that*

$$B_h(u, u) \geq C_0 \|\!|u\|\!|_h^2 \tag{92}$$

Proof: For any $u \in H^1_0$, we have

$$B_h(u, u) = |\Pi^0 u|^2_{0,I} + ((\varepsilon + d_p)u_x; u_x) + (\Pi^0 u; d_p u_x) \tag{93}$$

We have

$$\begin{aligned}
(\Pi^0 u; d_p u_x)_{0,I} &= \sum_{j=2}^{N} (\Pi^0 u; d_p u_x)_{0,K_{j-1/2}} && (94)\\
&\leq \sum_{j=2}^{N} h_{j-1/2} |\Pi^0 u|^2_{0,K_{j-1/2}} + \frac{1}{4h_{j-1/2}} |d_p u_x|^2_{0,K_{j-1/2}} && (95)\\
&\leq h|\Pi^0 u|^2_{0,I} + \frac{1}{8}(d_p u_x; u_x) && (96)
\end{aligned}$$

where we have used $d_{p|K_{j-1/2}} = h_{j-1/2} D_{p,j-1/2} \leq \frac{1}{2} h_{j-1/2}$. Therefore

$$B(u, u) \geq (1 - h)|\Pi^0 u|^2_{0,I} + ((\varepsilon + \frac{7}{8} d_p)u_x; u_x) \geq C_0 \|\!|u\|\!|_h^2 \tag{97}$$

■

The following estimates hold

Proposition 4.3 *Suppose that $u_{xx} \in L^\infty(I)$, where u is the solution of (1), then the solution $(u_h, p_h) \in P^1_{c,0} \times P^1_c$ of the box-scheme (83) verifies the a priori error estimates, (C is a generic constant independent of h),*

$$(i) \qquad \|\!|u - u_h\|\!|_h \leq Ch \max(\varepsilon^{1/2}; h^{1/2}) |u_{xx}|_{\infty,I} \tag{98}$$

$$(ii) \qquad |\Pi^0(p - p_h)|_{0,I} \leq Ch \max(\varepsilon; h) |u_{xx}|_{\infty,I} \tag{99}$$

$$(iii) \qquad |\Pi^0(p_x - p_{h,x})|_{0,I} \leq Ch^{1/2} \max(\varepsilon^{1/2}; h^{1/2}) |u_{xx}|_{\infty,I} \tag{100}$$

Proof: Let us begin by stating the standard interpolate estimate in the $\|\!|\ .\ \|\!|_h$ norm. If $\tilde{u} \in P^1_{c,0}$ is the standard interpolate of $u \in H^1_0(I)$, we have the error estimate

$$\begin{aligned} \|\!|u-\tilde{u}\|\!|_h^2 &= |\Pi^0(u-\tilde{u})|^2_{0,I} + |(\varepsilon + d_p)(u-\tilde{u})_x;(u-\tilde{u})_x|_{0,I} \\ &\leq |u-\tilde{u}|^2_{0,I} + (\varepsilon + h/2)|u-\tilde{u}|^2_{1,I} \\ &\leq Ch^4|u_{xx}|^2_{\infty,I} + (\varepsilon + h/2)h^2|u_{xx}|^2_{\infty,I} \\ &\leq Ch^2\max(\varepsilon,h)|u_{xx}|^2_{\infty,I} \end{aligned}$$

which gives,

$$\|\!|u-\tilde{u}\|\!|_h \leq Ch\max(\varepsilon^{1/2},h^{1/2})|u_{xx}|_{0,I} \tag{101}$$

The error estimate follows now a classical sketch. We denote by $E^p(x)$, $E^u(x)$ the two piecewise constant quadrature error functions defined for u, $p = -\varepsilon u_x$ by

$$(\Pi^0 p)_{j-1/2} = \frac{1}{2}(p(x_j) + p(x_{j-1}) - D_{p,j-1/2}(p(x_j) - p(x_{j-1})) - E^p_{j-1/2} \tag{102}$$

$$(\Pi^0 u)_{j-1/2} = \frac{1}{2}(u(x_j) + u(x_{j-1})) - E^u_{j-1/2} \tag{103}$$

We check easily that, if $u_{xx} \in L^\infty(I)$, the two following estimates hold, where C is a generic constant independent of h

$$|E^p|_{\infty,I} \leq Ch\varepsilon|u_{xx}|_{\infty,I} \qquad\qquad |E^u|_{\infty,I} \leq Ch^2|u_{xx}|_{\infty,I} \tag{104}$$

Moreover, we check easily that for any $v_h \in P^1_{c,0}$,

$$B_h(\tilde{u},v_h) = L_h(v_h) + M_h(v_h) \tag{105}$$

where the consistency error is

$$M_h(v_h) = (E^u;\Pi^0 v_h + d_p v_{h,x}) + (E^p;v_{h,x}) \tag{106}$$

Using Lemma (4.1), we have

$$C_0\|\!|\tilde{u}-u_h\|\!|_h^2 \leq B_h(\tilde{u}-u_h;\tilde{u}-u_h) \tag{107}$$

Now, denoting $\tilde{u} - u_h = v_h$, we have

$$B_h(\tilde{u}-u_h;\tilde{u}-u_h) = B_h(\tilde{u}-u_h,v_h) = L_h(v_h) + M_h(v_h) - L_h(v_h) = M_h(v_h) \tag{108}$$

Therefore, we deduce from (106) that

$$C_0\|\!|\tilde{u}-u_h\|\!|_h^2 \leq |(E^u;\Pi^0 v_h + d_p v_{h,x})| + |(E^p;v_{h,x})| \tag{109}$$

We observe now that, for any $v \in H^1_0(I)$ the two following estimates hold

$$|\Pi^0 v| \leq \|\!|v\|\!|_h \quad , \quad |v_x|_{0,I} \leq \frac{1}{\max(\varepsilon,\frac{C_m}{2}h)^{1/2}}\|\!|v\|\!|_h \tag{110}$$

This allows to estimate $|M_h(v_h)|$ by

$$
\begin{aligned}
|M_h(v_h)| &\le |E^u|_{\infty,I}(|\Pi^0 v_h|_{0,I} + \frac{h}{2}|v_{h,x}|_{0,I}) + |E^p|_{\infty,I}|v_{h,x}|_{0,I} \\
&\le Ch^2\{1 + C\frac{h}{\max(h^{1/2},\varepsilon^{1/2})}\}|u_{xx}|_{\infty,I}\|v_h\|_h + \frac{Ch\varepsilon}{\max(h^{1/2},\varepsilon^{1/2})}|u_{xx}|_{\infty,I}\|v_h\|_h \\
&\le Ch\max(h^{1/2},\varepsilon^{1/2})|u_{xx}|_{\infty,I}\|v_h\|_h
\end{aligned}
$$

Therefore

$$
\|\tilde{u} - u_h\|_h \le Ch\max(h^{1/2},\varepsilon^{1/2})|u_{xx}|_{\infty,I} \tag{111}
$$

Estimate (98) results now from (101) and (111).
For *(ii)*, we first check that $|\Pi^0 p - \Pi^0 p_h|_{0,I} \le |p - \Pi^0 p_h|_{0,I}$. Then, using formula (86) and $f = u + u_x - \varepsilon u_{xx}$, we have

$$
\begin{aligned}
p - \Pi^0 p_h &= (\varepsilon + d_p)u_{h,x} - d_p\Pi^0(f - u_h) - \varepsilon u_x \\
&= -(\varepsilon + d_p)(u_x - u_{h,x}) - d_p(\Pi^0 u_x - u_x) - d_p\Pi^0(u - u_h) + d_p\varepsilon\Pi^0(u_{xx})
\end{aligned}
$$

Therefore

$$
|\Pi^0 p - \Pi^0 p_h|_{0,I} \le Ch\max(\varepsilon,h)|u_{xx}|_{\infty,I} + Ch^2|u_{xx}|_{\infty,I} + Ch^2\max(\varepsilon^{1/2},h^{1/2})|u_{xx}|_{\infty,I} + Ch\varepsilon|u_{xx}|_{\infty,I}
$$

and finally (99).
For *(iii)*, we have

$$
\Pi^0 p_x - \Pi^0 p_{h,x} = \Pi^0 p_x - p_{h,x} = \Pi^0 p_x - (\Pi^0(f - u_h) - u_{h,x}) = -\Pi^0(u + u_x - u_h - u_{h,x})
$$

Therefore, using (110)

$$
\begin{aligned}
|\Pi^0 p_x - \Pi^0 p_{h,x}|_{0,I} &\le |\Pi^0(u - u_h)|_{0,I} + |u_x - u_{h,x}|_{0,I} \\
&\le \|u - u_h\|_h + \frac{1}{\max(\varepsilon, \frac{C_m}{2}h)^{1/2}}\|u - u_h\|_h \\
&\le C(1 + \frac{1}{\max(\varepsilon, \frac{C_m}{2}h)^{1/2}})h\max(\varepsilon^{1/2}; h^{1/2})|u_{xx}|_{\infty,I}
\end{aligned}
$$

which gives (100). ■

5 A fourth order box-scheme for the stationary convection-diffusion equation

5.1 Presentation of the scheme

Still starting from system (53), we use as in Sect. 3.2 the Euler-MacLaurin quadrature formula for approximating $(\Pi^0\, p)_{|j-1/2}$ on the box $K_{j-1/2}$. Recall that it reads

$$
\Pi^0 p_{j-1/2} = \frac{1}{2}(p(x_j) + p(x_{j-1})) - \frac{1}{12}h_{j-1/2}(p_x(x_j) - p_x(x_{j-1})) - E^p_{j-1/2} \tag{112}
$$

with the error term, (see (33))

$$E^p_{j-1/2} = \frac{B_4 h^4_{j-1/2}}{4!} p^{(4)}(\xi_{j-1/2}), \qquad x_{j-1} < \xi_{j-1/2} < x_j \tag{113}$$

Using the two relations $p_x(x) = f(x) - c u_x(x)$ and $u_x = -\frac{1}{\varepsilon} p(x)$, we get

$$\begin{aligned} \Pi^0 p_{j-1/2} &= \frac{1}{2}(p(x_j) + p(x_{j-1})) - \frac{1}{12}\frac{c h_{j-1/2}}{\varepsilon}(p(x_j) - p(x_{j-1})) \\ &- \frac{1}{12} h_{j-1/2}(f(x_j) - f(x_{j-1})) - E^p_{j-1/2} \end{aligned}$$

Replacing now in (53) $\Pi^0 p_{j-1/2}$ by the approximation

$$\Pi^0 p_{j-1/2} \sim \frac{1}{2}(p_j + p_{j-1}) - \frac{1}{12}\frac{c h_{j-1/2}}{\varepsilon}(p_j - p_{j-1}) - \frac{1}{12} h_{j-1/2}(f(x_j) - f(x_{j-1})) \tag{114}$$

allows to introduce the box-scheme

$$\left\{ \begin{array}{l} c[u_j - u_{j-1}] + [p_j - p_{j-1}] = h_{j-1/2}(\Pi^0 f)_{j-1/2} \\ (\frac{1}{2} - \frac{1}{12}\frac{c h_{j-1/2}}{\varepsilon}) p_j + (\frac{1}{2} + \frac{1}{12}\frac{c h_{j-1/2}}{\varepsilon}) p_{j-1} + \frac{\varepsilon}{h_{j-1/2}}(u_j - u_{j-1}) = \frac{1}{12} h^2_{j-1/2}(\Pi^0 f')_{j-1/2} \\ u_1 = u_N = 0 \end{array} \right. \tag{115}$$

If we define the coefficient $D_{p,j-1/2}$ as

$$D_{p,j-1/2} = \frac{1}{12}\frac{c h_{j-1/2}}{\varepsilon} = \frac{1}{6}\,\mathrm{sgn}(c) Pe_{j-1/2} \tag{116}$$

with definition of $Pe_{j-1/2}$ in (59), then the resolution in function of (p_{j-1}, p_j) yields

$$\begin{aligned} p_{j-1} &= -\left[\varepsilon - c\left(\frac{1}{2} - D_{p,j-1/2}\right) h_{j-1/2}\right] \Delta\, u_{j-1/2} \\ &- [\frac{1}{2} - D_{p,j-1/2}]\, h_{j-1/2}\, (\Pi^0 f)_{j-1/2} + \frac{1}{12} h^2_{j-1/2}(\Pi^0 f'))_{j-1/2} \\ p_j &= -\left[\varepsilon + c\left(\frac{1}{2} + D_{p,j-1/2}\right) h_{j-1/2}\right] \Delta\, u_{j-1/2} \\ &+ [\frac{1}{2} + D_{p,j-1/2}]\, h_{j-1/2}\, (\Pi^0 f)_{j-1/2} + \frac{1}{12} h^2_{j-1/2}(\Pi^0 f'))_{j-1/2} \end{aligned}$$

Identifying the two values of p_j in $K_{j-1/2}$, $K_{j+1/2}$ gives the scheme in $\Delta\, u_{j+1/2} = (u_{j+1} - u_j)/h_{j+1/2}$, $\Delta\, u_{j-1/2} = (u_j - u_{j-1})/h_{j-1/2}$.

$$\begin{aligned} &- \left\{ [\varepsilon + (D_{p,j+1/2} - 1/2)\, c\, h_{j+1/2}] \Delta\, u_{j+1/2} - [\varepsilon + (D_{p,j-1/2} + 1/2) c\, h_{j-1/2}] \Delta\, u_{j-1/2} \right\} \\ &= \left(\frac{1}{2} + D_{p,j-1/2}\right) h_{j-1/2}\, (\Pi^0 f)_{j-1/2} + \left(\frac{1}{2} - D_{p,j+1/2}\right) h_{j+1/2}\, (\Pi^0 f)_{j+1/2} \\ &+ \frac{1}{12} h^2_{j-1/2}(\Pi^0 f')_{j-1/2} - \frac{1}{12} h^2_{j+1/2}(\Pi^0 f')_{j+1/2} \end{aligned}$$

The most interesting feature of scheme (115), which is experimentally known for finite difference compact schemes, [6, 31], is that it is non oscillating, even when $\varepsilon \to 0^+$. For, consider the homogeneous case $f = 0$. Condition (60) reads here with (116)

$$\varepsilon + \frac{1}{12}\frac{c^2 h_{j-1/2}^2}{\varepsilon} \geq \frac{|c|}{2} h_{j-1/2} \tag{117}$$

The minimum of the left-hand side is obtained for $\varepsilon = \frac{|c| h_{j-1/2}}{2\sqrt{3}}$ giving a value of $\frac{1}{\sqrt{3}}|c| h_{j-1/2} \geq \frac{|c|}{2} h_{j-1/2}$. Therefore (117) is verified and the scheme is non-oscillating in the sense of Prop. 4.1 Finally, on an equally spaced mesh, the scheme in u_j, consistent with equation $cu_x - \varepsilon u_{xx} = f$ reads ($D_p = \frac{1}{12}\frac{ch}{\varepsilon}$)

$$\begin{aligned} &- \frac{1}{h}\left\{ [\varepsilon - ch(\frac{1}{2} - \frac{1}{12}\frac{ch}{\varepsilon})]\frac{u_{j+1} - u_j}{h} - [\varepsilon + ch(\frac{1}{2} + \frac{1}{12}\frac{ch}{\varepsilon})]\frac{u_j - u_{j-1}}{h} \right\} \\ &= (\frac{1}{2} + \frac{1}{12}\frac{ch}{\varepsilon})(\Pi^0 f)_{j-1/2} + (\frac{1}{2} - \frac{1}{12}\frac{ch}{\varepsilon})(\Pi^0 f)_{j+1/2} + \frac{1}{12}h^2(\Pi^0 f')_{j-1/2} - \frac{1}{12}h^2(\Pi^0 f')_{j+1/2} \end{aligned} \tag{118}$$

The scheme in p_j, consistent with $cp - \varepsilon p_x = -\varepsilon f$ is

$$\frac{1}{2}c(p_j + p_{j-1}) - (\frac{c^2 h}{12\varepsilon} + \frac{\varepsilon}{h})(p_j - p_{j-1}) = -\varepsilon(\Pi^0 f)_{j-1/2} + \frac{h^2}{12}(\Pi^0 f')_{j-1/2} \tag{119}$$

Performing an analysis similar to (66), we check easily that each of the schemes (118),(119) is fourth order accurate.

5.2 Comparison with the SUPG method

Similarly to the box-scheme in Sect. 4.1, we can recast the box-scheme (115) in the form of a non-standard SUPG scheme in the unknown $u_h \in P^1_{c,0}$, coupled to a local reconstruction formula for the flux $p_h \in P^1_c$. In addition, we obtain superconvergence results of order 4, between the standard interpolates $(\tilde{u}, \tilde{p}) \in P^1_{c,0} \times P^1_c$ of the solution, and the approximation (u_h, p_h). Let us define the piecewise constant function

$$\delta_h(x) = \sum_{j=2}^{N} h_{j-1/2}^2 \mathbb{1}_{K_{j-1/2}} \tag{120}$$

The following result holds

Proposition 5.1 *The box-scheme (115) is equivalent to:*
(i) the modified SUPG method for u_h : find $u_h \in P^1_{c,0}$ such that for any $v_h \in P^1_{c,0}$

$$(cu_{h,x} \ ; \ v_h + \frac{c}{12\varepsilon}\delta_h v_{h,x}) + \varepsilon\,(u_{h,x} \ ; \ v_{h,x}) = (\Pi^0 f \ ; \ v_h + \frac{c}{12\varepsilon}\delta_h v_{h,x}) + \frac{1}{12}(\Pi^0 f'; \delta_h v_{h,x}) \tag{121}$$

(ii) the local reconstruction formula in $p_h \in P^1_c$

$$p_{h|K_{j-1/2}} = (\Pi^0 p_h)_{j-1/2} + p_{h,x|K_{j-1/2}}(x - x_{j-1/2}) \tag{122}$$

with

$$(\Pi^0 p_h)_{j-1/2} = -(\varepsilon + \frac{c^2 h_{j-1/2}^2}{12\varepsilon})\, u_{h,x|K_{j-1/2}} + \frac{1}{12}h_{j-1/2}^2[\frac{c}{\varepsilon}(\Pi^0 f)_{j-1/2} + (\Pi^0 f')_{j-1/2}] \tag{123}$$

$$p_{h,x|K_{j-1/2}} = (\Pi^0 f)_{j-1/2} - cu_{h,x|K_{j-1/2}} \tag{124}$$

Proof: Follows the same lines as Prop. 4.2. ■

We conclude by some error estimates for the convection-diffusion problem (81), whose mixed form is

$$\left\{\begin{array}{l} u+u_x+p_x=f \\ p+\varepsilon u_x=0 \\ u\,(0)=u\,(1)=0 \end{array}\right. \tag{125}$$

The Euler-MacLaurin formula (31) yields for the two averages $(\Pi^0 u)_{j-1/2}$, $(\Pi^0 p)_{j-/1/2}$

$$\left\{\begin{array}{l} (\Pi^0 u)_{j-1/2}=\frac{1}{2}(u(x_j)+u(x_{j-1}))-\frac{h_{j-1/2}}{12}(u_x(x_j)-u_x(x_{j-1}))-E^u_{j-1/2} \\ (\Pi^0 p)_{j-1/2}=\frac{1}{2}(p(x_j)+p(x_{j-1}))-\frac{h_{j-1/2}}{12}(p_x(x_j)-p_x(x_{j-1}))-E^p_{j-1/2} \end{array}\right. \tag{126}$$

where the error terms are

$$E^u_{j-1/2}=\frac{B_4}{4!}h^4_{j-1/2}u^{(4)}(\xi^2_{j-1/2}) \quad , \quad E^p_{j-1/2}=\frac{B_4}{4!}h^4_{j-1/2}p^{(4)}(\xi^1_{j-1/2}) \tag{127}$$

Averaging $(125)_1$, $(125)_2$ on each box $K_{j-1/2}$ yields

$$\left\{\begin{array}{l} h_{j-1/2}(\Pi^0 u)_{j-1/2}+(u(x_j)-u(x_{j-1}))+(p(x_j)-p(x_{j-1}))=h_{j-1/2}(\Pi^0 f)_{j-1/2} \\ h_{j-1/2}(\Pi^0 p)_{j-1/2}=-\varepsilon(u(x_j)-u(x_{j-1})) \end{array}\right. \tag{128}$$

replacing $(\Pi^0 u)_{j-1/2}$, $(\Pi^0 p)_{j-1/2}$ by the fourth order approximation deduced from (126) gives the box-scheme

$$\left\{\begin{array}{l} \dfrac{1}{2}(u_j+u_{j-1})+(1+\dfrac{h^2_{j-1/2}}{12\varepsilon})\dfrac{p_j-p_{j-1}}{h_{j-1/2}}+\dfrac{u_j-u_{j-1}}{h_{j-1/2}}=(\Pi^0 f)_{j-1/2} \\ \dfrac{1}{2}(p_j+p_{j-1})-\dfrac{h_{j-1/2}}{12\varepsilon}(p_j-p_{j-1})+(\varepsilon+\dfrac{h^2_{j-1/2}}{12})\dfrac{u_j-u_{j-1}}{h_{j-1/2}}=\dfrac{h^2_{j-1/2}}{12}\Pi^0(f')_{j-1/2} \end{array}\right. \tag{129}$$

Still denoting $u_h\in P^1_{c,0}$, $p_h\in P^1_c$ defined by (19), and $\beta_h(x)=\sum_j(h^2_{j-1/2}/12\varepsilon)\mathbb{1}_{K_{j-1/2}}(x)$, (129) is proved to be equivalent to:

(i) find $u_h\in P^1_{c,0}$ solution of

$$\begin{aligned} & (\Pi^0 u_h+u_{h,x};\frac{1}{1+\beta_h}\{v_h+\beta_h v_{h,x}\})+(\varepsilon(1+\beta_h)u_{h,x};v_{h,x}) \\ = \; & (\Pi^0 f\;;\frac{1}{1+\beta_h}\{v_h+\beta_h v_{h,x}\})+\frac{1}{12}(\delta_h(\Pi^0 f');v_{h,x}) \qquad \forall v_h\in P^1_{c,0} \end{aligned} \tag{130}$$

(ii) reconstruct $p_h\in P^1_c$ by the local formula

$$p_{h|K_{j-1/2}}=(\Pi^0 p_h)_{j-1/2}+p_{h,x|K_{j-1/2}}(x-x_{j-1/2}) \tag{131}$$

where $p_{h,x|K_{j-1/2}}$, $(\Pi^0 p_h)_{j-1/2}$ are deduced from (129)

$$p_{h,x|K_{j-1/2}}=\frac{1}{1+\beta_h}[\Pi^0(f-u_h)-u_{h,x}] \tag{132}$$

$$(\Pi^0 p_h)_{j-1/2}=\beta_h p_{h,x}-\varepsilon(1+\beta_h)u_{h,x}+\frac{1}{12}\delta_h(\Pi^0 f')_{j-1/2} \tag{133}$$

Now, the box-scheme (129) can be rewritten

$$B_h(u_h, v_h) = L_h(v_h) \tag{134}$$

where the bilinear form $B_h(u_h, v_h)$ and the linear form $L_h(v_h)$ are respectiveley given by the left hand side and right hand side in (130). The energy norm is defined, for $u \in H_0^1$, by

$$|\!|\!| u |\!|\!|_h^2 = (\frac{1}{1+\beta_h}\Pi^0 u; \Pi^0 u) + ((\varepsilon(1+\beta_h) + \frac{\beta_h}{1+\beta_h}) u_x; u_x) \tag{135}$$

For any $u \in P^1_{c,0}$, we can check that $B_h(u,u) = |\!|\!| u |\!|\!|_h^2$. We state finally the two following *a priori* error estimates between u_h, p_h and the interpolants $\tilde{u}$, $\tilde{p}$ of u, p, whose proof follows the lines of Prop.4.3

Proposition 5.2 *If the solution u of (1) is sufficiently regular, then there exists $C > 0$, independent of h such that*

$$(i) \quad |\!|\!| \tilde{u} - u_h |\!|\!|_h \leq Ch^4(|u^{(4)}|_{\infty,I} + |u^{(5)}|_{\infty,I}) \tag{136}$$

$$(ii) \quad |\Pi^0 \tilde{p} - \Pi^0 p_h|_{0,I} \leq Ch^4(|u^{(4)}|_{\infty,I} + |u^{(5)}|_{\infty,I}) \tag{137}$$

6 Conclusion

In this paper, we derive two box-schemes for the stationary convection-diffusion equation following principles introduced in [22, 9, 10, 11, 12, 13]. The design introduced is basically of finite volume type, but results finally in a mixed Petrov-Galerkin scheme with an access by a local formula to the diffusive flux. The main intention of this paper is to fill a gap, at least for monodimensional problems, between the standard SUPG method, the mixed finite element method and the high order compact finite difference schemes. The extension to multidimensions is in progress.

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