A FINITE DIFFERENCE SCHEME FOR CONVECTION PROBLEMS ON THE SPHERE

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14ème colloque franco-roumain de mathématiques
30 août 2018

Collaborators and students

- M. Brachet (former doctoral student, IECL, Metz, now post-doc at INRIA Grenoble): Compact schemes on the Cubed Sphere.
- N. Paldor (Inst. Earth Sciences, Heb. Univ., Jerusalem): Linearized Shallow Water Equations on the sphere.
- M. Ben-Artzi, J. Falcovitz (Einstein Inst., Heb. Univ., Jerusalem): PDE's on the sphere, derivation of the spherical shallow water equations.
- B. Portelenelle (former master student, IECL, Metz, now doctoral student): Numerical quadrature on the Cubed Sphere.

Outline

1. The Cubed Sphere grid
2. Discrete differential geometry on the Cubed Sphere
3. Test cases for spherical convection problems

The Cubed Sphere grid

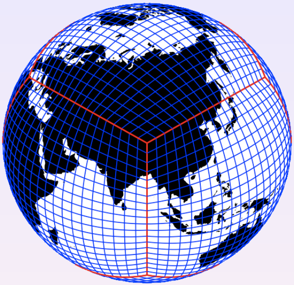


Figure: The Cubed Sphere.

Longitude/latitude grid

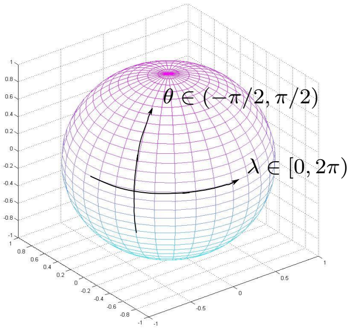
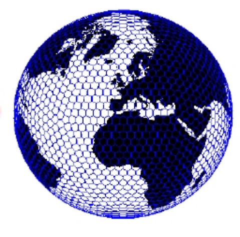


Figure: The lon-lat grid

Other grids



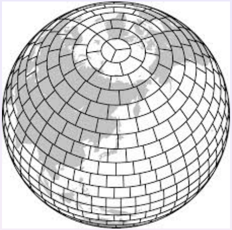
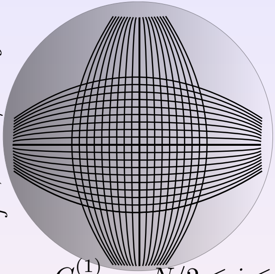


Figure: Left: icosahedral grid. Right: web grid.

The Cubed Sphere: two series of great circles



• Grid points are located at the intersection of two series of great circles. • The circles in *vertical* position are labeled $C_i^{(1)}$, $-N/2 \leq i \leq N/2$. • The circles in *horizontal* position are labeled $C_j^{(2)}$, $-N/2 \leq j \leq N/2$. • The integer N is a measure of the spatial resolution.

The Cubed Sphere grid (CS)

Panel on the Cubed Sphere

- Only the two circles in the center intersect orthogonally.
- With aperture angles $\widehat{C^{(1)}_{-N/2}, C^{(1)}_{N/2}} = \pi/2$ and $\widehat{C^{(2)}_{-N/2}, C^{(2)}_{N/2}} = \pi/2$, the grid forms a **panel**.
- The Cubed Sphere grid consists of six panels with number $k = (I), (II), (III), (IV), (V)$ and (VI).
- These six panels are labeled: *FRONT*, *EAST*, *BACK*, *WEST*, *NORTH* and *SOUTH*

Definition of the Cubed Sphere

The Cubed Sphere is the set of points $s^k_{i,j}$, $-N/2 \leq i, j \leq N/2$, $(I) \leq k \leq (VI)$.

The Cubed Sphere grid

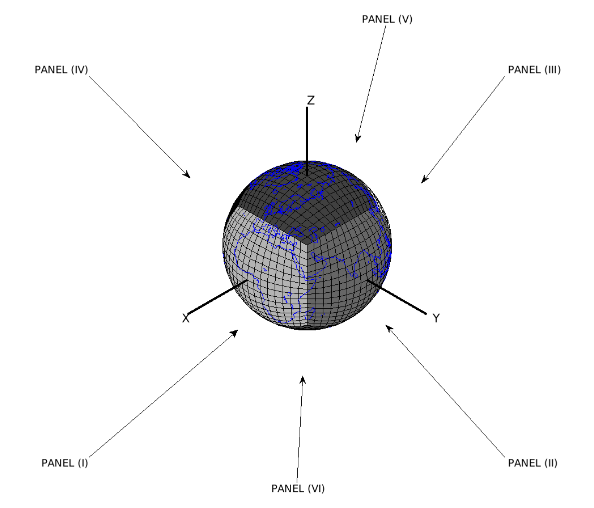
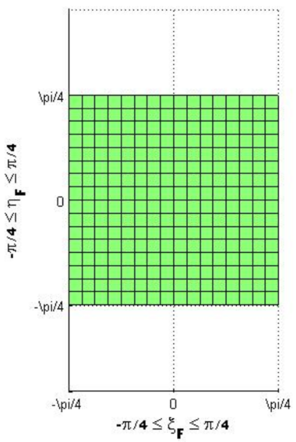


Figure: The Cubed Sphere grid with resolution $N = 16$ (16^2 cells by panel).

Coordinate angles on a panel

Angles ξ and η

- **First equatorial angle** ξ with $-\pi/4 \leq \xi \leq \pi/4$ ("zonal"),
- **Second equatorial angle** η with $-\pi/4 \leq \eta \leq \pi/4$ ("meridional").



Local basis

Local basis

- The local basis at $\boldsymbol{x} \in \mathbb{S}^2$ is ("covariant" basis):

$$(\boldsymbol{g}_\xi(\boldsymbol{x}), \boldsymbol{g}_\eta(\boldsymbol{x})) = (\partial_\xi \boldsymbol{x}, \partial_\eta \boldsymbol{x}) \tag{1}$$

- The dual basis is $(\boldsymbol{g}^\xi(\boldsymbol{x}), \boldsymbol{g}^\eta(\boldsymbol{x}))$:

$$\begin{cases} \boldsymbol{g}^\xi = G^{11}\boldsymbol{g}_\xi + G^{12}\boldsymbol{g}_\eta, \\ \boldsymbol{g}^\eta = G^{21}\boldsymbol{g}_\xi + G^{22}\boldsymbol{g}_\eta, \end{cases} \tag{2}$$

- The metric tensor G is

$$G \triangleq \begin{bmatrix} \boldsymbol{g}_\xi \cdot \boldsymbol{g}_\xi & \boldsymbol{g}_\xi \cdot \boldsymbol{g}_\eta \\ \boldsymbol{g}_\eta \cdot \boldsymbol{g}_\eta & \boldsymbol{g}_\eta \cdot \boldsymbol{g}_\eta \end{bmatrix} = \begin{bmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{bmatrix}^{-1} \tag{3}$$

- The coordinate lines $\xi =$ cste and $\eta =$ cste correspond to great circles.

Discrete data on the Cubed Sphere

Discrete data and great circles

A **grid function** on the Cubed Sphere consists of the six sets of data

$$u^k_{i,j}, \quad -N/2 \leq i,j \leq N/2, \quad k = (I), (II), (III), (IV), (V), (VI).$$

Arrangement of data along great circles

The data on the Cubed Sphere are partially arranged along great circles. The coordinate lines of each panel are great circles sections.

Example: two set of great circles associated with the panels FRONT and BACK

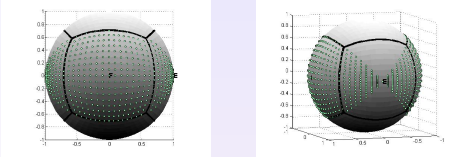


Figure: Set of ”horizontal” circles: iso $-\eta$ coordinate great circles.

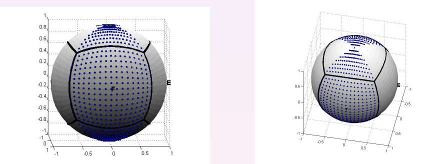


Figure: Set of ”vertical” circles: iso $-\xi$ coordinate great circles.

Discrete derivative

Compact Hermitian derivative

Hermitian discrete derivative $u_{x,j} \simeq u'(x_j)$ is defined by

$$\frac{1}{6}u_{x,j-1} + \frac{2}{3}u_{x,j} + \frac{1}{6}u_{x,j+1} = \frac{u_{j+1} - u_{j-1}}{2h} \qquad (4)$$

Fourth order accuracy

$$u_{x,j} = u'(x_j) - \frac{\mathbf{1}}{\mathbf{180}}\partial_x^{(5)}u(x_j)h^4 + O(h^6) \qquad (5)$$

Main idea

Calculate Hermitian derivatives along coordinate great circles on the Cubed Sphere.

Discrete spherical gradient

Continuous spherical gradient

Evaluation of the discrete gradient $\boldsymbol{\nabla}_T(\boldsymbol{x})u$ on panels FRONT and BACK is expressed as

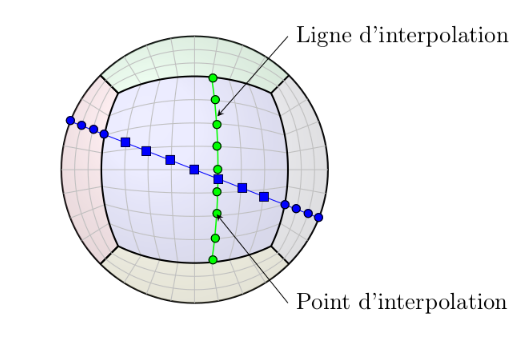
$$\boldsymbol{\nabla}_T u(\boldsymbol{x}) = \frac{\partial u}{\partial \xi}(\boldsymbol{x})_{|\eta}\boldsymbol{g}^{\xi}(\boldsymbol{x}) + \frac{\partial u}{\partial \eta}(\boldsymbol{x})_{|\xi}\boldsymbol{g}^{\eta}(\boldsymbol{x}) \tag{6}$$

Discrete spherical gradient

A discrete formula analog to (6) is

$$\boldsymbol{\nabla}_{T,h} u_{i,j} = \underbrace{u_{\xi,i,j}}_{\xi-\text{Hermitian deriv.}} \boldsymbol{g}^{\xi}_{i,j} + \underbrace{u_{\eta,i,j}}_{\eta-\text{Hermitian deriv.}} \boldsymbol{g}^{\eta}_{i,j} \tag{7}$$

Data along a great circle



The Hermitian discrete derivative is applied to the data in blue.

Divergence and Curl on the sphere

Discrete divergence and curl

- $\nabla_T \cdot \mathbf{u} = \dfrac{\partial \mathbf{u}}{\partial \xi}_{|\eta} \cdot \mathbf{g}^\xi + \dfrac{\partial \mathbf{u}}{\partial \eta}_{|\xi} \cdot \mathbf{g}^\eta \approx \nabla_{T,\Delta} \cdot \mathbf{u} = \mathbf{u}_\xi \cdot \mathbf{g}^\xi + \mathbf{u}_\eta \cdot \mathbf{g}^\eta$

- $\nabla_T \times \mathbf{u} = \dfrac{\partial \mathbf{u}}{\partial \xi}_{|\eta} \times \mathbf{g}^\xi + \dfrac{\partial \mathbf{u}}{\partial \eta}_{|\xi} \times \mathbf{g}^\eta \approx \nabla_{T,\Delta} \times \mathbf{u} = \mathbf{u}_\xi \times \mathbf{g}^\xi + \mathbf{u}_\eta \times \mathbf{g}^\eta$

where $(\mathbf{g}^\xi, \mathbf{g}^\eta)$ is the dual basis at (ξ_i, η_j) and h_ξ, h_η, $\mathbf{u}_\xi$ and $\mathbf{u}_\eta$ are the Hermitian derivatives at points (ξ_i, η_j) on each panel (k).

Discrete differential operators : error estimate

Consider a function $\boldsymbol{x} \in \mathbb{S}^2 \mapsto h(\boldsymbol{x})$. The gridfunction h^* is defined by $h^*(\boldsymbol{s}_{i,j}^k) = h(\boldsymbol{s}_{i,j}^k)$. $\Delta\xi = \Delta\eta = \pi/(2N)$ is the resolution parameter.

Proposition

Let $h : \mathbb{S}^2 \mapsto \mathbb{R}$ and $\mathbf{u} : \mathbb{S}^2 \mapsto \mathbb{TS}^2$ be regular functions. Then:

- Gradient error:

$$\|(\nabla_T h)^* - \nabla_{T,\Delta} h^*\|_\infty \leq \mathcal{O}\left(\Delta^3\right) \tag{8}$$

- Divergence error:

$$\|(\nabla_T \cdot \mathbf{u})^* - \nabla_{T,\Delta} \cdot \mathbf{u}^*\|_\infty \leq \mathcal{O}\left(\Delta^3\right) \tag{9}$$

- Curl error:

$$\|(\nabla_T \times \mathbf{u})^* - \nabla_{T,\Delta} \times \mathbf{u}^*\|_\infty \leq \mathcal{O}\left(\Delta^3\right) \tag{10}$$

The operator "$*$" stands for : "restricted to the Cubed Sphere".

Example: accuracy of the approximated curl

Approximate curl of a tangential vector field $\mathbf{u} = u(\theta)\mathbf{e}_\lambda$ (zonal dependance)

$$u(\theta) = \begin{cases} \frac{80}{e_n} \exp\left[\frac{1}{(\theta-\theta_0)(\theta-\theta_1)}\right] & \text{if } \theta_0 \leq \theta \leq \theta_1 \\ 0 & \text{else} \end{cases}$$

with e_n a normalization constant.

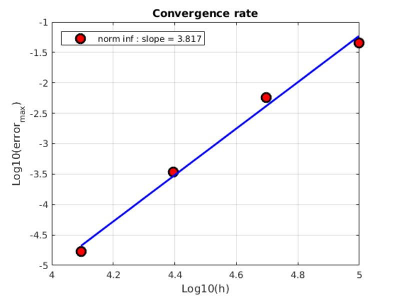


Figure: Convergence rate of the Hermitian curl of the zonal velocity $\mathbf{u} = u(\theta)\mathbf{e}_\lambda$.

PDE's on the sphere in climatology

- Atmospheric "horizontal" motion consists of the incompressible 3D Navier-Stokes equations (NS) in a spherical shell.
- The "vertical" motion consists of the thermodynamics (radiation, clouds, evaporation, etc).
- The "2D" model consists of the shallow water equations in nonlinear or linearized form. They are called SWE and LSWE. These models are obtained by averaging the NS equations along the vertical.
- Almost no theoretical results in the nonlinear regime !
- Even in the linear regime a lot remains to do !

Outline

- Center scheme in space of order 4 based on great circles on the Cubed Sphere.
- Minimal dissipation added by a filter function (10th order) AND the time scheme (RK4). Similar to Comp. Aeroacoustics.
- Compute numerical solutions after a physical time as long as possible using an eulerian scheme.
-
    - Short time: 1-10 days.
    - Medium time: 50-100 days.
    - Long time: 500-1000 days.
- Compare the quality of the solutions obtained with high order conservative methods.
- Prove mathematical results: conservation, convergence analysis, etc.

Time dependent PDE's on the sphere

Approximation in space

PDE at the continuous level: $(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{S}^2 \mapsto q(t, \boldsymbol{x})$ solution of

$$\partial_t q + \mathrm{div}_T \, \mathbf{f}(t, q) = 0 \tag{11}$$

is approximated by (method of lines):

- Approximation $q(t, \boldsymbol{s}_{i,j}^k) \simeq Q_{i,j}^k(t)$ solution of the ODE

$$\frac{d}{dt} Q(t) = F(t, Q(t)) \tag{12}$$

  where $F(t, Q)_{i,j}^k = -\,\mathrm{div}_{T,h} \, \mathbf{f}(t, Q)_{i,j}^k$
- Approximation in time of (12) is performed separately.

Time approximation and filtering

$$\frac{d}{dt}Q(t) = F(t, Q(t)) \tag{13}$$

is approximated by the RK4 time scheme. Let $\mathcal{F}$ be a spatial filter function.

Runge-Kutta order 4 + Spatial filtering

1. $K_1 = F(t^n, Q^n)$,
2. $K_2 = F(t^n + \frac{\Delta t}{2}, Q^n + \frac{\Delta t}{2} K_1)$,
3. $K_3 = F(t^n + \frac{\Delta t}{2}, Q^n + \frac{\Delta t}{2} K_2)$,
4. $K_4 = F(t^n + \Delta t, Q^n + \Delta t K_3)$
5. $\hat{Q}^{n+1} = Q^n + \frac{\Delta t}{6}(K_1 + 2K_2 + 2K_3 + K_4)$
6. $Q^{n+1} = \mathcal{F}(\hat{Q}^{n+1})$

Filtering step

High frequencies filtering *(Alpert, Tam, Visbal, Bailly, ...)*

The filter $\mathcal{F}$ is a discrete linear operator

$$\mathcal{F}(u)_i = \sum_{j=0}^{J} \frac{a_j}{2}(u_{i+j} + u_{i-j}) \tag{14}$$

The coefficients a_j satisfy:

- $$1 = \sum_{j=0}^{J} a_j, \quad \text{consistency} \tag{15}$$
- $$0 = \sum_{j=0}^{J} a_j(-1)^j \quad +1/-1 \text{ mode cancellation} \tag{16}$$
- $$0 = \sum_{j=0}^{J} \frac{a_j}{2} j^{2k} \text{ for } k = 2, 4.., 2(J-1) \quad \text{accuracy} \tag{17}$$

Filtering step (cont)

Proposition

There exists a unique filter function $\mathcal{F}$ satisfying the 3 conditions above:

Filtering step (cont.)

10-th order filtering

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 772/1024 \\ 210/1024 \\ -120/1024 \\ 45/1024 \\ -10/1024 \\ 1/1024 \end{pmatrix}$$

Symmetric filtering step

Symmetric filter in ξ and η:

$$\mathcal{F} = \frac{1}{2}\left(\mathcal{F}_\xi \circ \mathcal{F}_\eta + \mathcal{F}_\eta \circ \mathcal{F}_\xi\right)$$

Spherical Shallow Water equations

SW equations in non conservative form (vector form of the momentum equation)

$$\left\{ \begin{array}{rcl} \dfrac{\partial h^\star}{\partial t} + \nabla_T \cdot (h^\star \mathbf{v}) & = & 0 \\ \dfrac{\partial \mathbf{v}}{\partial t} + \nabla_T \left(\dfrac{1}{2} |\mathbf{v}|^2 + gh \right) + (f + \zeta)\, \mathbf{n} \times \mathbf{v} & = & 0 \end{array} \right. \quad (18)$$

where

- h is the fluid thickness and $\mathbf{v}$ the tangential velocity,
- $h^\star = h - h_s$ with h_s the bottom topography.
- $\mathbf{n}$ is the normal exterior vector,
- $\zeta = (\nabla_T \times \mathbf{v}) \cdot \mathbf{n}$ is the vorticity,
- f is the Coriolis parameter (depends on the latitude).

Conservation properties

Time invariant averaged values

Let $(h, \mathbf{v})$ be a solution of the SW equations, then the following quantities are conserved :

- **mass :** $\int_{\mathbb{S}_a^2} h(t,\mathbf{x}) d\sigma(\mathbf{x})$
- **energy :** $\int_{\mathbb{S}_a^2} \left(\frac{1}{2} g(h^2 - h_s^2) + \frac{1}{2} h |\mathbf{v}|^2 \right) d\sigma(\mathbf{x})$
- **potential enstrophy :** $\int_{\mathbb{S}_a^2} \frac{(f+\zeta)^2}{2gh} d\sigma(\mathbf{x})$

Quadrature over the Cubed Sphere

Discrete quadrature formula :

$$I(f) = a \Delta\xi \Delta\eta \sum_{k=(I)}^{(VI)} \sum_{i,j=-N/2}^{N/2} \sqrt{G_{i,j}^k} f_{i,j}^k \tag{19}$$

Barotropic instability test case

J. Galewsky and al., 2004

Zonal steady state of the SW equations

$$\begin{aligned} \bar{h}(\theta) &= h_0 + \frac{1}{g}\int_{-\pi/2}^{\theta} a u(\tau)\left[f + \frac{\tan(\tau)}{a}u(\tau)\right] d\tau \\ \mathbf{v}(\lambda,\theta) &= u(\theta)\mathbf{e}_\lambda \end{aligned} \qquad (20)$$

with :

- Coriolis parameter : $f = 2\Omega \sin\theta$,
- $u(\theta) = \begin{cases} \dfrac{u_{max}}{e_n} \exp\left(\dfrac{1}{(\theta-\theta_0)(\theta-\theta_1)}\right) & \text{if } \theta_0 \le \theta \le \theta_1 \\ 0 & \text{else} \end{cases}$

  with $e_n = C^{ste}$, $u_{max} = 80\,\mathrm{m\,s^{-1}}$, $\theta_0 = \pi/7$ and $\theta_1 = \pi/2 - \theta_0$.

Barotropic instability test case (cont.)

Perturbation of the steady state

The initial data is given by :

- initial h = perturbation of $\bar{h}$:

$$h(\lambda,\theta) = \bar{h}(\lambda,\theta) + \hat{h}\cos\theta \exp\left[-\left(\frac{\lambda}{\alpha}\right)^2 - \left(\frac{\theta_2-\theta}{\beta}\right)^2\right], \hat{h}/\bar{h} \approx 1\%$$

  with $\theta_2 = \pi/4$, $\alpha = 1/3$ and $\beta = 1/15$.

- zonal velocity not perturbed

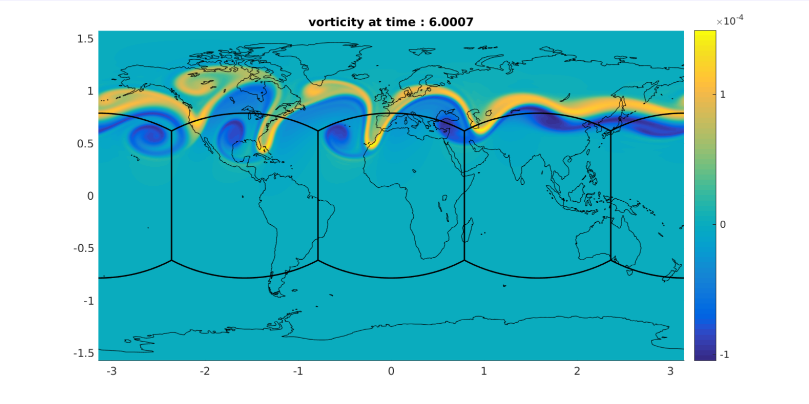
$$\mathbf{v}(\lambda,\theta) = u(\theta)\mathbf{e}_\lambda$$

Interest for the CS

This test is particularly challenging for the Cubed Sphere:

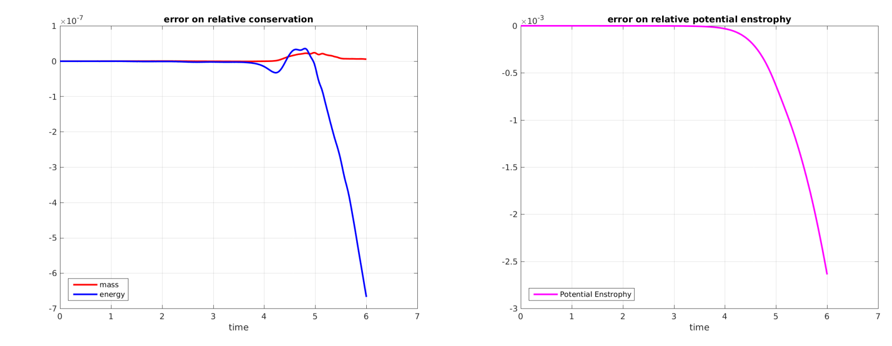
- h has large variations along the boundary of panel (V)
- the initial perturbation is located at the boundary between panel (I) and panel (V).

Barotropic instability test case (cont.)



- Vorticity after 6 days with a CS grid $6 \times 128 \times 128 \Rightarrow$ gives the correct number of vortices.
- Results similar to high order conservative methods such as FV or DG.

Numerical conservation



- Conservation of mass and energy, grid: $6 \times 128 \times 128$.
- Mass and energy are conserved up to 10^{-7} (relative error).
- Potential enstrophy is conserved up to 10^{-3} (relative error).

Rossby-Haurwitz test case

Williamson and al., 1992

Design of the test case

The Rossby-Haurwitz test case is an exact solution of the barotropic vorticity equation.

- initial velocity is $\mathbf{v} = u \cdot \mathbf{e}_\lambda + v \cdot \mathbf{e}_\theta$ with:

$$\left\{ \begin{array}{lcl} u & = & a\omega \cos\theta + aK \cos^{R-1}\theta \left(R \sin^2\theta - \cos^2\theta\right) \cos R\lambda \\ v & = & -aKR \cos^{R-1}\theta \sin\theta \sin R\lambda \end{array} \right. \tag{21}$$

- initial height h is

$$gh = gh_0 + a^2 A(\theta) + a^2 B(\theta) \cos R\lambda + a^2 C(\theta) \cos 2R\lambda \tag{22}$$

Rossby-Haurwitz test case (cont.)

Williamson and al., 1992

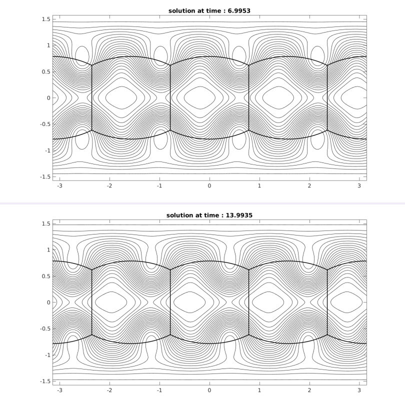


Figure: Numerical results of Rossby-Haurwitz wave test case with grid $80 \times 80 \times 6$ at 7 and 14 days. Contour line are plotted from 8100 m to 10500 m with interval of 100 m.

Rossby-Haurwitz test case (cont.)

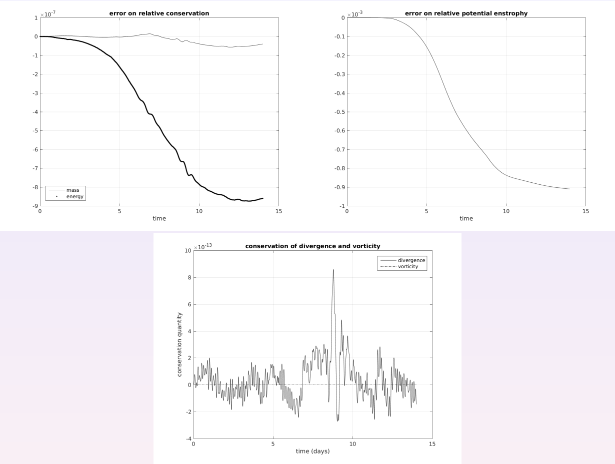


Figure: Conservation of Rossby-Haurwitz wave test case with grid $128 \times 128 \times 6$.

Linearized Shallow Water Equations (LSWE) on the rotating earth

LSWE equations

$$\begin{cases} \partial_t \boldsymbol{v}(\boldsymbol{x},t) + g\boldsymbol{\nabla}_T\eta + f\boldsymbol{k}\times\boldsymbol{v} = S_{\boldsymbol{v}} \\ \partial_t\eta(\boldsymbol{x},t) + H\,\mathrm{div}_T\,\boldsymbol{v} = S_\eta \\ \boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x}),\ \ \eta(\boldsymbol{x},0) = \eta_0(\boldsymbol{x}) \end{cases} \tag{23}$$

$$\begin{cases} \bullet\ g = \text{ gravity acceleration,} \\ \bullet\ H = \text{ mean thickness of the atmosphere,} \\ \bullet\ f = \text{Coriolis force} \end{cases} \tag{24}$$

The LSWE is the basic model for convection in climatology

- This is the reference model for linear waves on the rotating sphere.
- Recent reference: N. Paldor: *N. Paldor: Shallow Water Waves on the Rotating Earth, SpringerBriefs in Earth System Sciences, 2015*

Quasi-analytic traveling waves solutions of LSWE

Problem

Assessing numerical scheme by using realistic traveling wave solutions. Analytic or quasi-analytic.

LSWE theory for spherical waves

Laplace tidal equations published by Laplace in 1776. Since then a complete theory for LSWE on the sphere is still not available !

The barotropic vorticity equation (BVE)

- The simplest model for the atmosphere, (incompressible Euler equation on the sphere.)
- First numerically integrated by Charney, Fjørtoft and von Neumann in 1950 on the ENIAC.

$$\frac{D}{Dt}(\zeta + f) = 0, \quad \zeta = (\nabla \times \boldsymbol{v}).\boldsymbol{k} \quad f = \text{Coriolis force} \tag{25}$$

- 1930/1940: Rossby-Haurwitz wave analysis based on the linearized form of BVE using spherical harmonics by Rossby and Haurwitz.
- The Rossby-Haurwitz waves ARE NOT a solution of LSWE.

Paldor *et al.* test case for LSWE

- LSWE is expressed as

$$\partial_t \boldsymbol{q}(t,\boldsymbol{x}) = A\boldsymbol{q} \tag{26}$$

- "Zonal" traveling wave solution

$$\boldsymbol{q}(t,\boldsymbol{x}) = \hat{\boldsymbol{q}}(\theta)\exp(ik(\lambda - Ct)) \tag{27}$$

- $\hat{\boldsymbol{q}}(\theta)$ deduced from $\psi(\theta)$ from a second order equation

$$\psi''(\theta) + F_{\alpha,k,C}(\theta)\psi(\theta) = 0 \tag{28}$$

  with $\alpha = gH/(2\Omega a)^2$ and with B.C $\psi = 0$ at $\theta = \pm\pi/2$.
- The constant α determines the thickness of the atmosphere ("thick" or "thin").
- One obtains sequences of eigensolutions identified as
  - EIG or WIG mode (eastward or westward inertial-gravity) mode
  - Rossby mode

  They are quasi-analytic solutions to be compared with the numerical ones.

Hovmöller diagrams

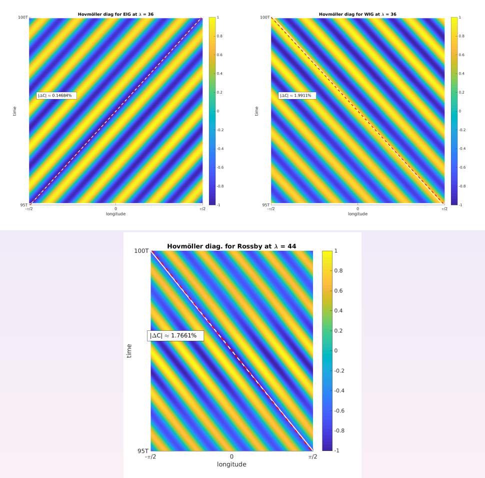


Figure: Hovmöller diagram for Paldor’s test cases: EIG solution, WIG solution and Rossby solution in the barotropic case

Paldor's test cases, (cont.)

With $N = 64$, one has the following data:

- EIG mode: 1807 iterations, 13.17 jours, error on the velocity C : 0.15%,
- WIG mode: 1807 iterations, 13.17 jours, error on the velocity C : 1.99%,
- Rossby mode : 165 142 iterations, 1203 jours, error on the velocity C : 1.77%.

Computational complexity

- Matlab sequential code on a deskwork (Intel(R) Xeon(R) CPU E5-2620 v2 @ 2.10GHz).
- Computational cost of $\nabla_T h \approx 96N^2$ for $12N^2$ unknowns (due to the tridiagonal matrices).
- Typical CPU time : 1.5 hours for 6 days with $N = 80$ $(2140$ it.).

Interest of the present approach

"Intrinsic" calculation

All finite difference approximations are obtained using great circles curvilinear abscissa.

Classic framework of finite-differences

- No pole problem, no "overlapping" grid problems.
- Thank to periodicity, easy change to another Hermitian formula without heavy recoding.
- Large backlog of numerical experiments from the Compact Scheme literature available.
- Cartesian Finite Difference more stable than spectral like collocation (Spherical Harmonics, Spherical Wavelets, Radial Basis Functions,...)
- To improve: avoid redundant calculations, higher order accuracy, implicit time stepping, fast solvers,...

Conclusion and future work

- Center scheme in space is relevant for spherical flows considered so far. Upwinding was found unnecessary.
- Cases considered so far are smooth, even if they are nonlinear.
- Minimal dissipation: explicit RK4 time scheme with filtering.
- Results similar to high order conservative schemes in terms of accuracy.
- Mathematical convergence analysis.

General references

- *On the Cubed Sphere:*
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